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Generalised Regularity and Compactness of Matrix Operators

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Abstract: A well–known result by Cohen and Dunford ([2], 1937) characterises the class of all regular compact linear operators. It follows that a regular matrix transformation cannot be compact. This means that if c denotes the set of all complex sequences of complex numbers, then an infinite matrix that maps c into c and preserves the limits cannot be compact. We obtained this result in a different way applying the theory of BK spaces from functional analysis and summability, and using the Hausdorff measure of noncompactness. Furthermore, we present the extension of this result to matrix transformations between the spaces c and the spaces of strongly summable sequences by the Cesàro method of order 1, and of strongly convergent sequences. We present new unified proofs for our main results.

Keywords: Sequence spaces, bounded linear operators, Hausdorff measure of noncompactness, regular and compact operators

1 Introduction and Notations

Measures of noncompactness are very useful tools in functional analysis, for instance in metric fixed point theory and the theory of operator equations in Banach spaces. They can also be used in the characterisations of classes of compact bounded operators between BK spaces by establishing identities or estimates for their Hausdorff measures of noncompactness. This approach was initiated on a large scale in [8], and later also presented in detail in [9, 10].

We consider the spaces of convergent sequences, of sequences that are strongly C_1–summable with index $p \geq 1$, and of strongly convergent sequences denoted by c, w^p and $[c]$, respectively, and apply the theory of BK spaces to characterise the classes of matrix operators from c, w^p and $[c]$ into c, and to determine their operator norms. Furthermore, we obtain the estimates for the Hausdorff measure of noncompactness of these operators, which yield the characterisations of the subclasses of compact matrix operators. Our main results are obtained by new unified results. They imply that matrix operators between those spaces that preserve the associated limits cannot be compact. They include the special case obtained by Cohen and Dunford [2].

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Now we list some of the used standard notations and results.

The set of all complex sequences $x = (x_k)_{k=1}^{\infty}$ is denoted by ω; we write ℓ_∞, c, c_0 and ϕ for the subsets of ω of all bounded, convergent, null and finite sequences, and ℓ_1 for the set of all absolutely convergent series.

Let $e = (e_k)_{k=1}^{\infty}$ and $e^{(n)} = (e_k^{(n)})_{k=1}^{\infty}$ for $n \in \mathbb{N}$ denote the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

A BK space X is a Banach sequence space with the property that all the coordinates $P_n : X \to \mathbb{C}$ with $P_n(x) = x_n$ $(x = (x_k)_{k=1}^{\infty} \in X)$ are continuous; a BK space X is said to have AK if $x = \lim_{m\to\infty} x^{[m]}$ for all $x = (x_k)_{k=1}^{\infty} \in X$, where $x^{[m]} = \sum_{k=1}^{m} x_k e^{(k)}$ denotes the m–section of the sequence $x = (x_k)_{k=1}^{\infty} \in X$.

Let $X, Y \subset \omega$ and $A = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix of complex entries.The β–dual of X is the set $X^\beta = \{a = (a_k)_{k=1}^{\infty} \in \omega : \sum_{k=1}^{\infty} a_k x_k$ converges for all $x = (x_k)_{k=1}^{\infty} \in X\}$. We write $A_n = (a_{nk})_{k=1}^{\infty}$ $(n \in \mathbb{N})$ and $A^k = (a_{nk})_{n=1}^{\infty}$ $(k \in \mathbb{N})$ for the sequences in the n^{th} row and the k^{th} column of A, $A_n x = \sum_{k=1}^{\infty} a_{nk} x_k$ for $n \in \mathbb{N}$ and $Ax = (A_n x)_{n=1}^{\infty}$ for the A–transform of the sequence x (provided all the series converge); $X_A = \{x \in \omega : Ax \in X$ for all $x \in X\}$ denotes the matrix domain of A in X, and (X, Y) is the class of all matrix transformations from X into Y, that is, $A \in (X, Y)$ if and only if $X \subset Y_A$, or equivalently, $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$.

Let X and Y be Banach spaces. Then we write, as usual, $\mathscr{B}(X, Y)$ for the Banach space of all bounded linear operators $L : X \to Y$ with the operator norm $\|L\| = \sup\{\|L(x)\| : \|x\| = 1\}$; if $Y = \mathbb{C}$, then $X^* = \mathscr{B}(X, \mathbb{C})$ denotes the continuous dual of X with the norm $\|f\| = \sup\{|f(x)| : \|x\| = 1\}$. The relation between the classes $\mathscr{B}(X, Y)$ and (X, Y) for BK spaces X and Y well–known and summarized in the following result.

Theorem 1.1 *Let X and Y be BK spaces.*
(a) ([12, Theorem 4.2.8]) *Then $(X, Y) \subset \mathscr{B}(X, Y)$, that is, every $A \in (X, Y)$ defines an operator $L_A \in \mathscr{B}(X, Y)$, where*

$$L_A(x) = Ax \text{ for all } x \in X. \tag{1}$$

(b) ([3, Theorem 1.9]) *If X has AK then $\mathscr{B}(X, Y) \subset (X, Y)$, that is, for each $L \in \mathscr{B}(X, Y)$, there exists a matrix $A \in (X, Y)$ such that*

$$Ax = L(x) \text{ for all } x \in X; \tag{2}$$

in this case we say that the matrix A represents the operator L.

2 The spaces w^p and $[c]$ and their β– and continuous duals

The sets w^p of *strongly C_1 summable sequences, with index* $p \geq 1$ and the sets $[c]$ of *strongly convergent sequences* were studied by Maddox [5], and Kuttner and Thorpe [4], respectively.

Throughout, let $p \geq 1$.

If $y=(y_k)_{k=1}^{\infty}\in\omega$, then we write $\Delta y_k=y_k-y_{k-1}$ with the convention $y_0=0$.

We write

$$w^p=\left\{x\in\omega:\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|x_k-\xi_{w^p}|^p=0 \text{ for some } \xi_{w^p}\in\mathbb{C}\right\}$$

and

$$[c]=\left\{x\in\omega:\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|\Delta(kx_k)-\xi_{[c]}|=0 \text{ for some } \xi_{[c]}\in\mathbb{C}\right\}.$$

The unique complex numbers ξ_{w^p} and $\xi_{[c]}$ are referred to as the w^p– and $[c]$–limits of the sequence x. We also use the notation ξ_c for the usual limit of the sequence $x\in c$.

If $X\in\{c,w^p,[c]\}$, then the norms $\|\cdot\|_X$ are defined by

$$\|x\|_X=\begin{cases}\|x\|_\infty=\sup_k|x_k| & \text{for } X=c\\ \sup_\nu\left(\frac{1}{2^\nu}\sum_{k=2^\nu}^{2^{\nu+1}-1}|x_k|^p\right)^{1/p} & \text{for } x=w^p\\ \sup_\nu\frac{1}{2^\nu}\sum_{k=2^\nu}^{2^{\nu+1}-1}|\Delta(kx_k)| & \text{for } x=[c].\end{cases}$$

If $X\in\{c,w^p,[c]\}$, then we write X_0 for the set of all $x\in X$ with $\xi_X=0$.

The following result is well known, where the case $X=c$ is standard. The cases $X=w^p$ and $X=[c]$ are [8, Proposition 3.44] and [6, Theorem 2.2].

Proposition 2.1 *Let $X\in\{c,w^p,[c]\}$. Then we have:*
(a) X and X_0 are BK spaces with the norm $\|\cdot\|_X$, and X_0 is a closed subspace of X;
(b) X_0 has AK;
(c) every sequence $x=(x_k)_{k=1}^{\infty}\in X$ has a unique representation

$$x=\xi_X e+\sum_{k=1}^{\infty}(x_k-\xi_X)e^{(k)}.$$

We introduce the following notations for the β–duals of our spaces. We write $\max_\nu$ and $\sum_\nu$ for the maximum and sum taken over all indices k from 2^ν to $2^{\nu+1}-1$ $(\nu=0,1,\dots)$, and put $\mathscr{W}_p=\{a\in\omega:\|a\|_{\mathscr{M}_p}<\infty\}$ and $\mathscr{C}=\{a\in\omega:\|a\|_{\mathscr{C}}<\infty\}$, where

$$\|a\|_{\mathscr{W}_p}=\begin{cases}\sum_{\nu=0}^{\infty}2^\nu\max_\nu|a_k| & (p=1)\\ \sum_{\nu=0}^{\infty}2^{\nu/p}\left(\sum_\nu|a_k|^q\right)^{1/q} & (1<p<\infty;\ q=p/(p-1)),\end{cases}$$

$$\|a\|_{\mathscr{C}}=\sum_{\nu=0}^{\infty}2^\nu\max_\nu\left|\sum_{j=k}^{\infty}\frac{a_j}{j}\right|;$$

we also use the notations

$$\mathscr{X} = \begin{cases} \ell_1 & (X = c) \\ \mathscr{W}_p & (X = w^p) \\ \mathscr{C} & (X = [c]) \end{cases} \quad \text{and } \|a\|_1 = \sum_{k=1}^{\infty} |a_k| \text{ for } a \in \ell_1.$$

Let $a \in \omega$ and X be a BK space. Then we write

$$\|a\|_X^* = \sup_{\|x\|=1} \left| \sum_{k=1}^{\infty} a_k x_k \right|,$$

provided the expression on the right exists and is finite, which is the case by [12, Theorem 7.2.9], whenever $a \in X^\beta$.

The following results are known, where the case $X = c$ is standard. The cases $X = w^p$ and $X = [c]$ are [8, Proposition 3.47] and [7, Theorem 2].

Proposition 2.2 *Let $X \in \{c, w^p, [c]\}$. Then we have:*
(a) $X^\beta = X_0^\beta = \mathscr{X}$, X_0^* and $\mathscr{X}$ are norm isomorphic, and $\|a\|_{w^p}^* = \|a\|_{\mathscr{X}}$ for all $x \in X$;
(b) $f \in X^*$ if and only if there exist $b \in \mathbb{C}$ and a sequence $a = (a_k)_{k=1}^{\infty} \in \mathscr{X}$ such that

$$f(x) = \xi_X b + \sum_{k=1}^{\infty} a_k x_k \text{ for all } x \in X,$$

where

$$a = \left(f(e^{(n)}) \right)_{n=1}^{\infty} \text{ and } b = f(e) - \sum_{n=1}^{\infty} a_n.$$

Moreover, if $f \in X^*$, then

$$\|f\| = |b| + \|a\|_{\mathscr{X}}. \tag{3}$$

3 Matrix transformations

In this section, we characterise the classes (X, Y) and (X_0, Y) for $X \in \{c, w^p, [c]\}$ and $Y \in \{c_0, c\}$.

Theorem 3.1 *Let $X \in \{c, w^p, [c]\}$. Then*
(a) $A \in (X_0, c)$ if and only if

$$\|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_{\mathscr{X}} < \infty \tag{4}$$

and

$$\alpha_k = \lim_{n\to\infty} a_{nk} \text{ exists for each } k; \tag{5}$$

$A \in (X_0, c_0)$ *if and only if the condition in* (4) *holds and*

$$\lim_{n\to\infty} a_{nk} = 0 \text{ for each } k; \tag{6}$$

(b) $A \in (X, c)$ *if and only if the conditions in* (4) *and* (5) *hold and*

$$\tilde{\alpha} = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} \text{ exists}; \tag{7}$$

$A \in (X, c_0)$ *if and only if the conditions in* (4) *and* (6) *hold and*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 0. \tag{8}$$

(c) If $A \in (X, Y)$ *for* $Y \in \{c_0, c\}$*, then*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)}, \tag{9}$$

(with L_A *from* (1)*).*

Proof. Since X_0 and and X are BK spaces by Proposition 2.1 (a), it follows from [10, Theorem 9.3.3 (c) (9.10)] that $A \in (Z, \ell_\infty)$ for $Z = X_0$ and $Z = X$ if and only if

$$\|A\|_{(Z,\ell_\infty)} = \sup_n \|A_n\|_Z^* < \infty. \tag{10}$$

We also have $\|\cdot\|_Z^* = \|\cdot\|_{\mathscr{X}}$ by Proposition 2.2 (a), so it follows from (10) that $A \in (Z, \ell_\infty)$ if and only if the condition in (4) is satisfied.
(a) Now let $Z = X_0$, a BK space with AK by Proposition 2.1 (b). Since c and c_0 are closed subspaces of the BK space ℓ_∞, we obtain from [12, 8.3.6] that $A \in (X_0, Y)$ for $Y = c$ and $Y = c_0$ if and only if the condition in (4) holds in both cases, and $A^k \in Y$ for each k, which yields the conditions in (5) for $Y = c$ and in (6) for $Y = c_0$.
(b) Since $X = X_0 \oplus e$, it follows from [12, 8.3.7] that $A \in (X, Y)$ for $Y = c$ or $Y = c_0$ if and only if $A \in (X_0, Y)$ and $Ae \in Y$, which accounts for the additional conditions in accounts in (7)for $Y = c$ and (8).
(c) Part c follows from [10, Theorem 9.3.3 (c) (9.11)]. □

Remark 3.2 *(a) The case* $X = c$ *of* Theorem 3.1 (b) *is the famous classical* Toeplitz theorem [11] *(1911).*
(b) Since X_0 *has AK,* Theorem 3.1 *also characterises the classes of operators L in* $\mathscr{B}(X_0, c)$ *and* $\mathscr{B}(X_0, c_0)$ *and provides a formula for the operator norm* $\|L\|$*, where A is the matrix that represents L as in* (2).

4 The Hausdorff measure of noncompactness and compact operators

In this section, we give estimates for the Hausdorff measure of noncompactness of the operators L_A of the matrices $A \in (X,c)$, when $X \in \{c, w^p, [c]\}$, characterise the classes $\mathscr{K}(X,c)$ of compact matrix operators in (X,c), and apply our results to obtain that matrices in (X,c) that preserve the limits cannot be compact matrix operators.

It is useful to recall the definitions of the Hausdorff measures of noncompactness of bounded sets in complete metric spaces and of operators between Banach spaces.

Definition 4.1 ([1, Definition 2.1]) Let X be a complete metric space and $\mathscr{M}_X$ denote the class of all bounded subsets of X. Then the function $\chi : \mathscr{M}_X \to [0,\infty)$ with

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{–net in } X\}$$

is called the *Hausdorff* or *ball measure of noncompactness*.

Definition 4.2 ([8, Definition 2.24]) Let X and Y be Banach spaces and χ be the Hausdorff measure of noncompactness.
(a) An operator $L : X \to Y$ is said to be *χ–bounded*, if

$$L(Q) \in \mathscr{M}_Y \text{ for all } Q \in \mathscr{M}_X,$$

and if there exists a nonnegative real number C such that

$$\chi(L(Q)) \leq C \cdot \chi(Q) \text{ for all } Q \in \mathscr{M}_X. \tag{11}$$

(b) If an operator L is χ–bounded, then the number

$$\|L\|_\chi = \inf\{C \geq 0 : (11) \text{ holds}\}$$

is called the *Hausdorff measure of noncompactness of L.*

We need the following important, well–known results.

Theorem 4.3 (Goldenštein, Go'hberg, Markus) ([8, Theorem 2.23])
Let X be a Banach space with a Schauder basis $(b_n)_{n=1}^\infty$ and the operator $\mathscr{R}_n : X \to X$ for each $n \in \mathbb{N}$ be defined by

$$\mathscr{R}_n(x) = \sum_{k=n+1}^\infty \lambda_k b_k \text{ for all } x = \sum_{k=1}^\infty \lambda_k b_k \in X.$$

We put

$$\mu(Q) = \limsup_{n\to\infty} \left(\sup_{x\in Q} \|\mathscr{R}_n(x)\| \right) \text{ for all } Q \in \mathscr{M}_X.$$

Then the following inequalities hold for all $Q \in \mathcal{M}_X$

$$\frac{1}{a} \cdot \mu(Q) \leq \chi(Q) \leq \inf_n \left(\sup_{x \in Q} \|\mathcal{R}_n(x)\| \right) \leq \mu(Q), \tag{12}$$

where $a = \limsup_{n\to\infty} \|\mathcal{R}_n\|$ *is the* basis constant of the Schauder basis.

Theorem 4.4 *Let X and Y be BK spaces and* $L \in \mathcal{B}(X,Y)$.
Then we have
(a) ([8, Theorem 2.25])

$$\|L\|_\chi = \chi(L(S_X)) = \chi(L(\bar{B}_X)) = \chi(L(B_X)), \tag{13}$$

where S_X, $\bar{B}_X$ *and* B_X *denote the unit sphere, and the open and closed unit balls in X;*
(b) ([8, Corollary 2.26 (2.58)])

$$\|L\|_\chi = 0 \text{ if and only if } L \in \mathcal{K}(X,Y). \tag{14}$$

Now we establish an estimate for the Hausdorff measure of noncompactness of operators L_A for $A \in (X,c)$ when $X \in \{c, w^p, [c]\}$.

Theorem 4.5 *(a) Let* $X \in \{c, w^p, [c]\}$ *and* $A \in (X,c)$. *Then*

$$\frac{1}{2}\limsup_{n\to\infty}\left(\left|\sum_{k=1}^{\infty}\alpha_k - \tilde{\alpha}\right| + \|\tilde{A}_n\|_{\mathcal{X}}\right) \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left(\left|\sum_{k=1}^{\infty}\alpha_k - \tilde{\alpha}\right| + \|\tilde{A}_n\|_{\mathcal{X}}\right), \tag{15}$$

where L_A *is the operator as in* (1)*, the numbers* $\tilde{\alpha}$ *and* α_k $(k \in \mathbb{N})$ *are from* (7) *and* (5)*, and* $\tilde{A} = (\tilde{a}_{nk})_{n,k=1}^{\infty}$ *is the matrix with* $\tilde{a}_{nk} = a_{nk} - \alpha_k$ *for all n and k.*
(b) If $A \in (X,c)$*, then* $L_A \in \mathcal{K}(X,c)$ *if and only if*

$$\lim_{n\to\infty}\left(\left|\sum_{k=1}^{\infty}\alpha_k - \tilde{\alpha}\right| + \|\tilde{A}_n\|_{\mathcal{X}}\right) = 0. \tag{16}$$

Proof. We write $\|\cdot\| = \|\cdot\|_X$, for short.
Let $A \in (X,c)$. Then the numbers $\tilde{\alpha}$ and α_k $(k \in \mathbb{N})$ exist by Theorem 3.1 (c). Also, since X and c are BK spaces, $L_A \in \mathcal{B}(X,Y)$ by Theorem 1.1 (a).
(a) First we show

$$(\alpha_k)_{k=1}^{\infty} \in \mathcal{X}. \tag{17}$$

Since $X^\beta = X_0^\beta = \mathcal{X}$ by Proposition 2.2 (a), it suffices to show that $(\alpha_k)_{k=1}^{\infty} \in X_0^\beta$. It follows from the fact that X_0 has AK by Proposition 2.1 (b) that there exists a positive constant C such that $\|x^{[m]}\| \leq C\|x\|$, hence

$$\left|\sum_{k=1}^{m} a_{nk}x_k\right| = \left|A_n x^{[m]}\right| \leq \|A_n\|_X^* \|x^{[m]}\| \leq C\|A\|_{(X,\ell_\infty)}\|x\| \text{ for all } m \text{ and } n,$$

and so by (5) and (4)

$$\left|\sum_{k=1}^{m} \alpha_k x_k\right| = \lim_{n\to\infty}\left|\sum_{k=1}^{m} a_{nk}x_k\right| \le C\|A\|_{(X,\ell_\infty)}\|x\| < \infty \text{ for all } m,$$

that is, the sequence of partial sums $\sum_{k=1}^{m} \alpha_k x_k$ is bounded for each $x \in X_0$. Since X_0 has AK this implies $(\alpha_k)_{k=1}^{\infty} \in X_0^{\beta}$ by [12, Theorem 7.2.7 (iii)].

Now we show

$$\lim_{n\to\infty} A_n x = \sum_{k=1}^{\infty} \alpha_k x_k \text{ for all } x \in X_0. \tag{18}$$

Since $A \in (X_0, c)$ and $(\alpha_k)_{k=1}^{\infty} \in \mathscr{X}$ by (17), it follows for all n that

$$\|\tilde{A}_n\|_{\mathscr{X}} \le \|A_n\|_{\mathscr{X}} + \|(\alpha_k)_{k=1}^{\infty}\|_{\mathscr{X}} \le \|A\|_{(X,\ell_\infty)} + \|(\alpha_k)_{k=1}^{\infty}\|_{\mathscr{X}},$$

hence the matrix $\tilde{A}$ satisfies the condition in (4). Also we have $\lim_{n\to\infty} \tilde{a}_{nk} = 0$ for each k, that is, the matrix $\tilde{A}$ satisfies the condition in (6). Consequently $\tilde{A} \in (X_0, c_0)$ by Theorem 3.1 (a). Since $(\alpha_k)_{k=1}^{\infty} \in X_0^{\beta}$, we obtain (18).

Now let $x \in X$ and $y = Ax$. Then $x^{(0)} - \xi_X e \in X_0$. Since $A_n \in X^{\beta}$ for all n, and $e \in X$, we have

$$y_n = A_n x = A_n(x^{(0)} - \xi_X e) = A_n x^{(0)} + \xi A_n e \text{ for all } n,$$

and it follows from (18), (7) and (5) that

$$\eta = \lim_{n\to\infty} A_n x = \sum_{k=1}^{\infty} \alpha_k x_k^{(0)} + \xi_k \lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk} = \sum_{k=1}^{\infty} \alpha_k x_k + \xi_X\left(\tilde{\alpha} - \sum_{k=1}^{\infty}\alpha_k\right). \tag{19}$$

Now we have for all m

$$\mathscr{R}_m(y) = \sum_{n=m+1}^{\infty} (y_n - \eta)e^{(n)}.$$

We write $f_n^{(m)}(x) = ((\mathscr{R}_m \circ L_A)(x))_n$ for all n and m, and obtain for $n \ge m+1$ by (19)

$$f_n^{(m)}(x) = A_n x - \sum_{k=1}^{\infty} \alpha_k x_k + \xi_X\left(\sum_{k=1}^{\infty}\alpha_k - \tilde{\alpha}\right) = \tilde{A}_n x + \xi_X\left(\sum_{k=1}^{\infty}\alpha_k - \tilde{\alpha}\right).$$

Since $f_n^{(m)} \in c^*$, it follows by (3) that

$$\sup_{x\in S_X} \|\mathscr{R}_m(L_A(x))\|_\infty = \sup_{n\ge m+1} \|f_n^{(m)}\| = \sup_{n\ge m+1}\left(\left|\sum_{k=1}^{\infty}\alpha_k - \tilde{\alpha}\right| + \|\tilde{A}_n\|_{\mathscr{X}}\right).$$

Finally it follows from (12), (13) and the fact that $a = \limsup_{m\to\infty}\|\mathscr{R}_m\| = 2$ by [10, Example 7.9.7]

$$\frac{1}{2}\inf_m \|f_n^{(m)}\| \quad = \quad \frac{1}{2}\inf_m\left(\sup_{n\ge m+1}\left|\sum_{k=1}^{\infty}\alpha_k - \tilde{\alpha}\right| + \|\tilde{A}_n\|_{\mathscr{X}}\right)$$

$$= \frac{1}{2}\limsup_{m\to\infty}\left(\left|\sum_{k=1}^{\infty}\alpha_k - \tilde{\alpha}\right| + \|\tilde{A}_m\|_{\mathscr{X}}\right) \leq \|L_A\|_\chi \leq \inf_m \|f_n^{(m)}\|.$$

This yields the inequalities in (15).
(b) Part (b) is an immediate consequence of (15) and (14). □

Now we generalise the concept of regularity to that of *X–regularity* and apply Theorem 4.5 to conclude that an X–regular matrix transformation cannot be compact.

Again let $X \in \{c, w^p, [c]\}$. A matrix $A \in (X,c)$ is said to be *X–regular*, if $\lim_{n\to\infty} A_n x = \xi_X$ for all $x \in X$, where ξ_X is the X–limit of the sequence x. If $X = c$, then X–regularity is the usual regularity.

Corollary 4.6 *An X–regular matrix operator cannot be compact.*

Proof. Let $A \in (X,c)$ be X–regular. Then it follows from (19) that

$$\lim_{n\to\infty} A_n e^{(k)} = \lim_{n\to\infty} a_{nk} = \alpha_k = 0 \text{ for each } k, \text{ and } \lim_{n\to\infty} A_n e = \tilde{\alpha} = 1,$$

hence $1 + \|A_n\|_{\mathscr{X}} \geq 1$ for all n, and so (16) is not satisfied. □

Remark 4.7 *For $X = c$,* Corollary 4.6 *reduces to the known result of Cohen and Dunford* [2] *that a regular matrix transformation cannot be compact.*

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