

Second order complex differential equations with analytic coefficients in the unit disc

Mohamed Amine Zemirni, Benharrat Belaïdi

Abstract: In this article, we investigate the growth of solutions of second order complex differential equations in which the coefficients are analytic in the unit disc with lower $[p, q]$ -order. We've proved similar results as in the case of complex differential equations in the whole complex plane with usual $[p, q]$ -order. We define also new type of order applied on the coefficients to study the growth of solutions.

Keywords: Complex differential equation, analytic function, $[p, q]$ -order.

1 Introduction and main results

Nevanlinna theory has appeared to be powerful tool in the field of *complex differential equations*. First research in this field was started by H. Wittich and his students in the 1950's and 1960's, see [22]. After that many authors have investigated the complex differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_1(z)f' + A_0(z)f = 0 \quad (1)$$

and achieved many valuable results when the coefficients $A_0(z), A_1(z), \dots, A_{k-1}(z)$ are *entire* functions. The theory of complex differential equations in the unit disc has been developed since 1980's, see [19]. In the year 2000, Heittokangas in [9] firstly investigated the growth and oscillation theory of equation (1) when the coefficients $A_0(z), A_1(z), \dots, A_{k-1}(z)$ are *analytic* functions in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ by introducing the definition of the function spaces and his results also gave some important tools for further investigations on the theory of meromorphic solutions of equations (1). After that, many articles (see e.g. [1, 2, 3, 10, 15, 16, 21]) focused on this topic. In this article, we continue to focus on the same topic by considering the second order complex differential equation

$$f'' + A(z)f' + B(z)f = 0 \quad (2)$$

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Mohamed Amine Zemirni and Benharrat Belaïdi are with the Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (UMAB), B. P. 227 Mostaganem, Algeria.

when $A(z)$ and $B(z)$ are analytic functions in the unit disc \mathbb{D} .

Throughout this article, we will use the standard notations and fundamental results of the Nevanlinna value distribution theory of meromorphic functions, for more details on Nevanlinna theory and its applications in *complex differential equations* in complex plane and in unit disc, we refer to [8, 9, 14, 15, 24].

Before discussing the previous results and before we state our main results, we recall definitions and preliminary remarks concerning meromorphic and analytic functions in \mathbb{D} .

For a meromorphic function f in the unit disc \mathbb{D} , the *order of growth* of f is defined by

$$\sigma(f) := \limsup_{r \rightarrow 1^-} \frac{\log^+ T(r, f)}{\log \frac{1}{1-r}}$$

and the *lower order* of f is defined by

$$\mu(f) := \liminf_{r \rightarrow 1^-} \frac{\log^+ T(r, f)}{\log \frac{1}{1-r}}.$$

Here, $T(r, f)$ is the Nevanlinna characteristic function of f which is expressed as follows

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(e^{i\varphi})| d\varphi + \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

where $\log^+ x := \max\{0, \log x\}$ ($x \geq 0$), $n(r, f)$ denotes the number of poles of f in $\{z : |z| \leq r\}$ and each pole counted according to its multiplicity. By definitions of the order and the lower order of growth of a meromorphic function f , it is clear that $\mu(f) \leq \sigma(f)$ holds in general. A meromorphic function for which order and lower order are the same is said to be of *regular growth*, and the meromorphic function which is not of regular growth is said to be of *irregular growth*.

We start with a result due to Gundersen.

Theorem 1.1 ([6]). *Let $A(z)$ and $B(z)$ be two entire functions with $\rho(A) < \rho(B)$. Then every nontrivial solution of the equation (2) is of infinite order.*

Here, $\rho(f)$ denotes the order of growth of f in complex plane which is defined by

$$\rho(f) := \limsup_{r \rightarrow +\infty} \frac{\log T(r, f)}{\log r}.$$

Heittokangas, modified a reasoning due to Gundersen in the complex plane to get the following result.

Theorem 1.2 ([9]). *Let $A(z)$ and $B(z)$ be two analytic functions with $\sigma(A) < \sigma(B)$. Then every nontrivial solution of the equation (2) is of infinite order.*

Recently, Long, Heittokangas and Ye, in [18], gave a similar result to Theorem 1.1 when the usual orders are replaced with the corresponding lower orders, and they proved the following theorem.

Theorem 1.3 ([18]). *Let $A(z)$ and $B(z)$ be two entire functions with $\mu(A) < \mu(B)$. Then every nontrivial solution of the equation (2) is of infinite order.*

Here, $\mu(f)$ denotes the order of growth of f in complex plane which is defined similarly as the order but for “liminf” instead of “limsup”.

Remark 1.4. *It is mentioned in [5, p. 238] that for any fixed μ and ρ satisfying $0 \leq \mu \leq \rho \leq \infty$ there exists an entire function with order ρ and lower order μ . Hence, in Theorem 1.1 and Theorem 1.3, if $\rho(A) = \rho(B)$ with A is of irregular growth and B is of regular growth, or, $\mu(A) = \mu(B)$ with A is of regular growth and B is of irregular growth, then we easily conclude that every nontrivial solution of the equation (2) is of infinite order.*

Now, we ask a natural question as follows : Is it possible to obtain the same conclusions on the solutions in the unit disc as in Theorem 1.3 by replacing the usual orders with lower orders ? In this article, we will discuss and answer on this question and other questions that will be mentioned later. So, the main purpose of this article is to investigate growth of solutions of the differential equation (2) under conditions on the coefficients in using lower order; actually, we show that possibly the similar results can be obtained when the usual orders are replaced with lower orders. In fact, we will prove our main results in using the general definitions of order and lower order that are the $[p, q]$ -order and lower $[p, q]$ -order. For that, we need to recall the following definitions and notations. Let us define inductively for $r \in [0, +\infty)$, $\exp_0 r := r$, $\exp_1 r := e^r$ and $\exp_{n+1} r := \exp(\exp_n r)$, $n \in \mathbb{N}$. For all r sufficiently large, we define $\log_0 r := r$, $\log_1 r := \log r$ and $\log_{n+1} r := \log(\log_n r)$, $n \in \mathbb{N}$. Moreover, we denote by $\exp_{-1} r := \log r$ and $\log_{-1} r := \exp_1 r$.

In [11, 12] Juneja, Kapoor and Bajpai have investigated some properties of entire functions of $[p, q]$ -order, lower $[p, q]$ -order and obtained some results about their growth. In [17], in order to maintain accordance with general definitions of the entire function f of iterated p -order [13, 14], Liu, Tu and Shi gave a minor modification of the original definition of the $[p, q]$ -order given in [11, 12]. Further, in [1, 2], Belaïdi defined $[p, q]$ -order of analytic and meromorphic functions in unit disc \mathbb{D} . For conveniences, we list the following concepts.

Definition 1.5 ([1, 2]). *Let $p \geq q \geq 1$, and f be a meromorphic function in \mathbb{D} . Then, the $[p, q]$ -order of f is given by*

$$\sigma_{[p,q]}(f) := \limsup_{r \rightarrow 1^-} \frac{\log_p^+ T(r, f)}{\log_q \frac{1}{1-r}}.$$

For an analytic function f in \mathbb{D} , we also define

$$\sigma_{M,[p,q]}(f) := \limsup_{r \rightarrow 1^-} \frac{\log_{p+1}^+ M(r, f)}{\log_q \frac{1}{1-r}},$$

where $M(r, f) = \max\{|f(z)| : |z| = r\}$.

Definition 1.6 ([10, 21]). Let $p \geq q \geq 1$, and f be a meromorphic function in \mathbb{D} . Then, the lower $[p, q]$ -order of f is given by

$$\mu_{[p,q]}(f) := \liminf_{r \rightarrow 1^-} \frac{\log_p^+ T(r, f)}{\log_q \frac{1}{1-r}}.$$

For an analytic function f in \mathbb{D} , we also define

$$\mu_{M,[p,q]}(f) := \liminf_{r \rightarrow 1^-} \frac{\log_{p+1}^+ M(r, f)}{\log_q \frac{1}{1-r}}.$$

Definition 1.7 ([11]). A function for which $[p, q]$ -order and lower $[p, q]$ -order are the same is said to be of regular $[p, q]$ -growth, and the function which is not of regular growth is said to be of irregular $[p, q]$ -growth.

Definition 1.8 ([10, 21]). Let $p \geq q \geq 1$, and f be a meromorphic function in \mathbb{D} with $[p, q]$ -order $0 < \sigma_{[p,q]}(f) < \infty$. Then, $[p, q]$ -type of f is given by

$$\tau_{[p,q]}(f) := \limsup_{r \rightarrow 1^-} \frac{\log_{p-1}^+ T(r, f)}{(\log_{q-1} \frac{1}{1-r})^{\sigma_{[p,q]}(f)}}.$$

For an analytic function f in \mathbb{D} , we also define the $M - [p, q]$ -type of f with $M - [p, q]$ -order $0 < \sigma_{M,[p,q]}(f) < \infty$ by

$$\tau_{M,[p,q]}(f) := \limsup_{r \rightarrow 1^-} \frac{\log_p^+ M(r, f)}{(\log_{q-1} \frac{1}{1-r})^{\sigma_{M,[p,q]}(f)}}.$$

Definition 1.9 ([10]). Let $p \geq q \geq 1$, and f be a meromorphic function in \mathbb{D} with lower $[p, q]$ -order $0 < \mu_{[p,q]}(f) < \infty$. Then, lower $[p, q]$ -type of f is given by

$$\underline{\tau}_{[p,q]}(f) := \liminf_{r \rightarrow 1^-} \frac{\log_{p-1}^+ T(r, f)}{(\log_{q-1} \frac{1}{1-r})^{\mu_{[p,q]}(f)}}.$$

For an analytic function f in \mathbb{D} , we also define the lower $M - [p, q]$ -type of f with lower $M - [p, q]$ -order $0 < \mu_{M,[p,q]}(f) < \infty$ by

$$\underline{\tau}_{M,[p,q]}(f) := \liminf_{r \rightarrow 1^-} \frac{\log_p^+ M(r, f)}{(\log_{q-1} \frac{1}{1-r})^{\mu_{M,[p,q]}(f)}}.$$

Remark 1.10. *It is easy to see that $0 \leq \sigma_{[p,q]}(f) \leq +\infty$. If f is non-admissible, i.e., $T(r, f) = O(\log \frac{1}{1-r})$, then $\sigma_{[p,q]}(f) = 0$ for any $p \geq q \geq 1$. We note that $\sigma_{[1,1]}(f) = \sigma(f)$ (order of growth), $\sigma_{[2,1]}(f) = \sigma_2(f)$ (hyper-order) and $\sigma_{[p,1]}(f) = \sigma_p(f)$ (iterated p -order).*

Proposition 1.11. *Let f be an analytic function in \mathbb{D} of $[p, q]$ -order. Then, the following statements hold :*

- (i) *If $p = q = 1$, then $\sigma(f) \leq \sigma_M(f) \leq \sigma(f) + 1$, see [20, p. 205].*
- (ii) *If $p = q \geq 2$ and $\sigma_{[p,q]}(f) < 1$, then $\sigma_{[p,q]}(f) \leq \sigma_{M,[p,q]}(f) \leq 1$, see [21].*
- (iii) *If $p = q \geq 2$ and $\sigma_{[p,q]}(f) \geq 1$, or $p > q \geq 1$, then $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f)$, see [1, 21].*

Similarly, we can get the following proposition.

Proposition 1.12. *Let f be an analytic function in \mathbb{D} of $[p, q]$ -order. Then, the following statements hold :*

- (i) *If $p = q = 1$, then $\mu(f) \leq \mu_M(f) \leq \mu(f) + 1$.*
- (ii) *If $p = q \geq 2$ and $\mu_{[p,q]}(f) < 1$, then $\mu_{[p,q]}(f) \leq \mu_{M,[p,q]}(f) \leq 1$.*
- (iii) *If $p = q \geq 2$ and $\mu_{[p,q]}(f) \geq 1$, or $p > q \geq 1$, then $\mu_{[p,q]}(f) = \mu_{M,[p,q]}(f)$.*

Latreuch and Belaidi in [16], see also [4, Lemma 3.7], proved the following theorem.

Theorem 1.13 ([4]). *Let $p \geq q \geq 1$ be integers. Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be analytic functions in \mathbb{D} satisfying*

$$\max \{ \sigma_{[p,q]}(A_j) : j = 1, \dots, k-1 \} < \sigma_{[p,q]}(A_0).$$

Then every nontrivial solution f of (1) satisfies $\sigma_{[p,q]}(f) = +\infty$ and

$$\sigma_{[p,q]}(A_0) \leq \sigma_{[p+1,q]}(f) \leq \max \{ \sigma_{M,[p,q]}(A_j) : j = 0, \dots, k-1 \}.$$

Furthermore, if $p > q$ then

$$\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0).$$

Tu and Huang in [21] proved the following theorem when the dominant coefficient is A_0 with lower $[p, q]$ -order instead of usual $[p, q]$ -order.

Theorem 1.14 ([21]). *Let $p \geq q \geq 1$ be integers. Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be analytic functions in \mathbb{D} satisfying*

$$\max \{ \sigma_{M,[p,q]}(A_j) : j = 1, \dots, k-1 \} < \mu_{M,[p,q]}(A_0).$$

Then every nontrivial solution f of (1) satisfies $\mu_{[p+1,q]}(f) = \mu_{M,[p,q]}(A_0)$.

In the next, we consider the second-order complex differential equation (2), and we show that the similar conclusions can be made when the usual orders in previous theorems are replaced with lower orders. In fact, we prove the following result.

Theorem 1.15. *Let $A(z)$ and $B(z)$ be two analytic functions in \mathbb{D} with $\mu_{[p,q]}(A) < \mu_{[p,q]}(B)$. Then, every nontrivial solution f of the equation (2) satisfies the following statements :*

- (i) *If $p = q = 1$, then $\sigma(f) = +\infty$ and $\sigma_2(f) \geq \mu_2(f) \geq \mu(B)$.*
- (ii) *If $p \geq q \geq 2$, then $\sigma_{[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) \geq \mu_{[p+1,q]}(f) \geq \mu_{[p,q]}(B)$.*

Remark 1.16. *The statement (i) in Theorem 1.15, is considered as a completion of the Theorems 1.1, 1.2 and 1.3.*

Remark 1.17. *From Theorems 1.2, 1.13 and 1.15 and by Definition 1.7, if $\sigma_{[p,q]}(A) = \sigma_{[p,q]}(B)$ with A is of irregular $[p,q]$ -growth and B is of $[p,q]$ -regular growth, or, $\mu_{[p,q]}(A) = \mu_{[p,q]}(B)$ with A is of regular $[p,q]$ -growth and B is of irregular $[p,q]$ -growth, then we easily conclude that every nontrivial solution f of the equation (2) is of infinite $[p,q]$ -order and $\sigma_{[p+1,q]}(f) \geq \mu_{[p,q]}(B)$.*

Belaïdi and Latreuch in [4], Tu and Huang in [21] used the concept of $[p,q]$ -type to investigate the growth of solutions of equation (2), they proved the following theorems.

Theorem 1.18 ([4, Lemma 3.11]). *Let $p \geq q \geq 1$ be integers, and let $A(z)$ and $B(z)$ be two analytic functions in \mathbb{D} with $\sigma_{[p,q]}(A) = \sigma_{[p,q]}(B) > 0$ and $0 < \tau_{[p,q]}(A) < \tau_{[p,q]}(B) < +\infty$. Then every nontrivial solution f of (2) satisfies $\sigma_{[p,q]}(f) = +\infty$ and*

$$\sigma_{[p,q]}(B) \leq \sigma_{[p+1,q]}(f) \leq \max \{ \sigma_{M,[p,q]}(A) ; \sigma_{M,[p,q]}(B) \}.$$

Furthermore, if $p > q$ then

$$\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(B).$$

Theorem 1.19 ([21]). *Let $p \geq q \geq 1$ be integers. Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be analytic functions in \mathbb{D} satisfying*

$$\max \{ \sigma_{M,[p,q]}(A_j) : j = 1, \dots, k-1 \} \leq \sigma_{M,[p,q]}(A_0) < +\infty$$

and

$$\max \{ \tau_{M,[p,q]}(A_j) : j = 1, \dots, k-1 \} < \tau_{M,[p,q]}(A_0).$$

Then every nontrivial solution f of (1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{M,[p,q]}(A_0)$.

Hu and Zheng in [10], used both the lower $[p,q]$ -order and lower $[p,q]$ -type on the dominant coefficient $A_0(z)$, and obtained the following theorem.

Theorem 1.20 ([10]). *Let p, q be integers such that $p > q \geq 2$, and $A_{k-1}(z), \dots, A_1(z), A_0(z) \not\equiv 0$ be analytic functions in \mathbb{D} with $0 < \mu = \mu_{[p,q]}(A_0) \leq \sigma_{[p,q]}(A_0) < \infty$. Assume that*

$$\max \{ \sigma_{[p,q]}(A_j) : j = 1, \dots, k-1 \} \leq \mu_{[p,q]}(A_0)$$

and that

$$\max \{ \tau_{[p,q]}(A_j) : \sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_0), j = 1, \dots, k-1 \} < \tau_{[p,q]}(A_0) = \tau < \infty.$$

If $f \not\equiv 0$ is a solution of (1), then we have $\mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0) \leq \sigma_{[p,q]}(A_0) = \sigma_{[p+1,q]}(f)$.

According to Theorem 1.15, the following question is naturally asked. What happen when $\mu_{[p,q]}(A) = \mu_{[p,q]}(B)$ in Theorem 1.15 ? to answer on this question, we prove the following theorem.

Theorem 1.21. *Let $p \geq q \geq 1$ be integers and let $A(z)$ and $B(z)$ be two analytic functions in \mathbb{D} with $0 < \mu_{[p,q]}(A) = \mu_{[p,q]}(B) < \infty$ and $0 < \tau_{[p,q]}(A) < \tau_{[p,q]}(B) < \infty$. Then, every non-trivial solution f of the equation (2) satisfies $\sigma_{[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) \geq \mu_{[p+1,q]}(f) \geq \mu_{[p,q]}(B)$.*

Remark 1.22. *It should be mentioned that Wu and Zheng in [23] gave some results about growth of solutions of (1) in using lower p -iterated order and lower p -iterated type. So, Theorem 1.21 generalizes Theorem 3.1 in [23], for second-order differential equation to $[p, q]$ -order.*

Now, the question is : What can be said about the growth of solution f of (2) when $\sigma_{[p,q]}(A) = \sigma_{[p,q]}(B)$ and $\tau_{[p,q]}(A) = \tau_{[p,q]}(B)$, or $\mu_{[p,q]}(A) = \mu_{[p,q]}(B)$ and $\tau_{[p,q]}(A) = \tau_{[p,q]}(B)$?

Hamouda in [7], to study the growth of meromorphic solutions of differential equations with finite p -iterated order in complex plane, introduced new type of growth (see [7, p. 46]) and obtained an interesting result ([7, Theorem 1.13]). According to the definition of this new type of growth, we introduce a new definition of type of growth that we note $\tau_{[p,q]}^*(f)$ related to $[p, q]$ -growth of meromorphic function f in the unit disc, as follows.

Definition 1.23. *For $1 \leq q \leq p$, let f be a meromorphic function of finite $[p, q]$ -order in \mathbb{D} such that $0 < \sigma_{[p,q]}(f) = \sigma < \infty$ and $0 < \tau_{[p,q]}(f) = \tau < \infty$, we define $\tau_{[p,q]}^*(f)$ by*

$$\tau_{[p,q]}^*(f) = \limsup_{r \rightarrow 1^-} \frac{\log_{p-2}^+ T(r, f)}{\exp\left(\tau \left(\log_{q-1} \frac{1}{1-r}\right)^\sigma\right)}.$$

By the same way, we define this new type of order in lower case $\tau_{[p,q]}^*(f)$ for a meromorphic function f where $0 < \mu_{[p,q]}(f) = \mu < \infty$ and $0 < \tau_{[p,q]}(f) = \tau < \infty$ by

$$\tau_{[p,q]}^*(f) = \liminf_{r \rightarrow 1^-} \frac{\log_{p-2}^+ T(r, f)}{\exp\left(\tau \left(\log_{q-1} \frac{1}{1-r}\right)^\mu\right)}.$$

For an analytic function f in \mathbb{D} such that $0 < \sigma_{M,[p,q]}(f) = \sigma_M < \infty$ and $0 < \tau_{M,[p,q]}(f) = \tau_M < \infty$, we also define $\tau_{M,[p,q]}^*(f)$

$$\tau_{M,[p,q]}^*(f) = \limsup_{r \rightarrow 1^-} \frac{\log_{p-1}^+ M(r, f)}{\exp\left(\tau_M \left(\log_{q-1} \frac{1}{1-r}\right)^{\sigma_M}\right)},$$

and for an analytic function f in \mathbb{D} such that $0 < \mu_{M,[p,q]}(f) = \mu_M < \infty$ and $0 < \underline{\tau}_{M,[p,q]}(f) = \underline{\tau}_M < \infty$, we also define $\underline{\tau}_{M,[p,q]}^*(f)$

$$\underline{\tau}_{M,[p,q]}^*(f) = \liminf_{r \rightarrow 1^-} \frac{\log_{p-1}^+ M(r, f)}{\exp\left(\underline{\tau}_M \left(\log_{q-1} \frac{1}{1-r}\right)^{\mu_M}\right)}.$$

Remark 1.24. In the case $p = 1$, we replace \log_{-1} by \exp , see page 3.

Example 1.25. We give these two examples to illustrate the Definition 1.23.

1. Let f be analytic function in \mathbb{D} defined by :

$$f(z) = \exp\left(2 \exp\left(4 \left(\log \frac{1}{1-z}\right)^5\right)\right).$$

It is clear that :

$$\sigma_{M,[2,2]}(f) = 5, \quad \tau_{M,[2,2]}(f) = 4 \text{ and } \tau_{M,[2,2]}^*(f) = 2.$$

2. We can deal similarly with case $p = 1$, as shown in this example. Let f be analytic function in \mathbb{D} defined by :

$$f(z) = 3 \exp\left(\frac{2}{1-z}\right).$$

It's clear that :

$$\sigma(f) = 1, \quad \tau(f) = \frac{2}{\pi} \text{ and } \tau^*(f) = 3.$$

By using this new concept, we will prove the following results.

Theorem 1.26. Let $p \geq 2$ and $1 \leq q \leq p$. Suppose that the analytic coefficients of the equation (2) satisfy

$$0 < \sigma_{[p,q]}(A) = \sigma_{[p,q]}(B) = \sigma < \infty,$$

$$0 < \tau_{[p,q]}(A) = \tau_{[p,q]}(B) = \tau < \infty$$

and

$$0 < \tau_{[p,q]}^*(A) < \tau_{[p,q]}^*(B) = \tau^* < \infty.$$

Then every solution $f \neq 0$ of the equation (2) satisfies $\sigma_{[p,q]}(f) = +\infty$ and

$$\sigma_{[p,q]}(B) \leq \sigma_{[p+1,q]}(f) \leq \max \{ \sigma_{M,[p,q]}(A); \sigma_{M,[p,q]}(B) \}.$$

Furthermore, if $p > q$ then

$$\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(B).$$

Example 1.27. In this example, we give a perspective to how Theorem 1.26 works. Let $A(z) = \exp(\exp(2(1-z)^{-1}))$ and $B(z) = \exp(\frac{1}{2}\exp(2(1-z)^{-1}))$ be two analytic functions in \mathbb{D} . We see here that $\sigma_2(A) = \sigma_2(B)$, $\tau_2(A) = \tau_2(B)$ and $\tau_2^*(A) < \tau_2^*(B)$. Then, it is not easy to solve exactly the equation $f'' + A(z)f' + B(z)f = 0$, and we cannot deduce anything about the growth of solutions by using previous results. But, by Theorem 1.26, we say that, every solution $f \neq 0$ of $f'' + A(z)f' + B(z)f = 0$ has $\sigma_2(f) = +\infty$ and $\sigma_3(f) = \sigma_2(B) = 1$.

Theorem 1.28. Let $p \geq 2$ and $1 \leq q \leq p$. Suppose that the analytic coefficients of the equation (2) satisfy

$$0 < \mu_{[p,q]}(A) = \mu_{[p,q]}(B) = \mu < \infty,$$

$$0 < \underline{\tau}_{[p,q]}(A) = \underline{\tau}_{[p,q]}(B) = \underline{\tau} < \infty$$

and

$$0 < \underline{\tau}_{[p,q]}^*(A) < \underline{\tau}_{[p,q]}^*(B) = \underline{\tau}^* < \infty.$$

Then every solution $f \neq 0$ of the equation (2) satisfies

$$\sigma_{[p,q]}(f) = +\infty, \text{ and } \sigma_{[p+1,q]}(f) \geq \mu_{[p,q]}(B).$$

2 Some Lemmas

Lemma 2.1 ([8, 9, 20]). Let f be a meromorphic function in the unit disc \mathbb{D} and let $k \in \mathbb{N}$. Then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

where $S(r, f) = O(\log^+ T(r, f) + \log(\frac{1}{1-r}))$, possibly outside a set $F \subset [0, 1)$ with finite logarithmic measure $\int_F \frac{dr}{1-r} < \infty$.

Remark 2.2. In this paper, we use several times the sets E which are not the same each time, although they are denoted by the same letter.

Lemma 2.3 ([10]). *Let f be a meromorphic function in \mathbb{D} with $\mu_{[p,q]}(f) = \mu < \infty$. Then for any given $\varepsilon > 0$, there exists a subset $E \subset [0, 1)$ that has an infinite logarithmic measure $\int_E \frac{dr}{1-r} = +\infty$ such that for all $r \in E$ we have*

$$T(r, f) < \exp_p \left\{ (\mu + \varepsilon) \log_q \left(\frac{1}{1-r} \right) \right\}.$$

Lemma 2.4. *Let f be a meromorphic function in \mathbb{D} with $0 < \mu_{[p,q]}(f) = \mu < \infty$ and $0 < \tau_{[p,q]}(f) = \tau < \infty$. Then, for any given $\varepsilon > 0$, there exists a set $E \subset [0, 1)$ that has an infinite logarithmic measure $\int_E \frac{dt}{1-t} = +\infty$ such that*

$$T(r, f) < \exp_{p-1} \left\{ (\tau + \varepsilon) \left(\log_{q-1} \frac{1}{1-r} \right)^\mu \right\}$$

holds for all $r \in E$.

Proof. By the definition of lower $[p, q]$ -order and lower $[p, q]$ -type, there exists an increasing sequence $\{r_m\}_{m=1}^{+\infty} \subset [0, 1)$ satisfying $1 - d(1 - r_m) < r_{m+1}$, ($0 < d < 1$), $(r_m \xrightarrow{m \rightarrow +\infty} 1^-)$ and

$$\lim_{m \rightarrow +\infty} \frac{\log_{p-1}^+ T(r_m, f)}{\left(\log_{q-1} \frac{1}{1-r_m} \right)^\mu} = \tau.$$

For any $r \in \left[1 - \frac{1-r_m}{d}, r_m\right]$, we have

$$\frac{\log_{p-1}^+ T(r, f)}{\left(\log_{q-1} \frac{1}{1-r} \right)^\mu} \leq \frac{\log_{p-1}^+ T(r_m, f)}{\left(\log_{q-1} \frac{1}{1-r_m} \right)^\mu} \frac{\left(\log_{q-1} \frac{1}{1-r_m} \right)^\mu}{\left(\log_{q-1} \frac{1}{1-r} \right)^\mu} \xrightarrow{m \rightarrow +\infty} \tau,$$

because $\left(\log_{q-1} \frac{1}{1-r} \right)^\mu \underset{m \rightarrow \infty}{\sim} \left(\log_{q-1} \frac{1}{1-r_m} \right)^\mu$. Then, for any given $\varepsilon > 0$, there exists a positive integer m_0 such that for all $m \geq m_0$ and for all $r \in \left[1 - \frac{1-r_m}{d}, r_m\right]$, we have

$$\log_{p-1}^+ T(r, f) < (\tau + \varepsilon) \left(\log_{q-1} \frac{1}{1-r} \right)^\mu.$$

Set

$$E = \bigcup_{m=m_0}^{+\infty} \left[1 - \frac{1-r_m}{d}, r_m\right].$$

Then, for all $r \in E$, we obtain for any given $\varepsilon > 0$,

$$T(r, f) < \exp_{p-1} \left\{ (\tau + \varepsilon) \left(\log_{q-1} \frac{1}{1-r} \right)^\mu \right\},$$

where

$$\int_E \frac{dt}{1-t} = \sum_{m=m_0}^{+\infty} \int_{1-\frac{1-r_m}{d}}^{r_m} \frac{dt}{1-t} = \sum_{m=m_0}^{+\infty} \log \frac{1}{d} = +\infty.$$

□

Lemma 2.5. *Let $p \geq 2$ and $1 \leq q \leq p$ and f be a meromorphic function in \mathbb{D} such that $0 < \sigma_{[p,q]}(f) = \sigma < \infty$, $0 < \tau_{[p,q]}(f) = \tau < \infty$ and $0 < \tau_{[p,q]}^*(f) = \tau^* < \infty$. Then for any given $\beta < \tau^*$, there exists a subset $E \subset [0, 1)$ that has an infinite logarithmic measure $\int_E \frac{dr}{1-r} = +\infty$ such that for all $r \in E$ we have*

$$\log_{p-2} T(r, f) > \beta \exp \left(\tau \left(\log_{q-1} \frac{1}{1-r} \right)^\sigma \right).$$

Proof. By the definitions, there exists an increasing sequence $\{r_m\}_{m=1}^{+\infty} \subset [0, 1)$ satisfying $\frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m < r_{m+1}$, $(r_m \xrightarrow{m \rightarrow +\infty} 1^-)$ and

$$\lim_{m \rightarrow +\infty} \frac{\log_{p-2} T(r_m, f)}{\exp \left(\tau \left(\log_{q-1} \frac{1}{1-r_m} \right)^\sigma \right)} = \tau^*.$$

Then, there exists a positive integer m_1 such that for all $m \geq m_1$ and for any given $0 < \varepsilon < \tau^*$, we have

$$\log_{p-2} T(r_m, f) > (\tau^* - \varepsilon) \exp \left(\tau \left(\log_{q-1} \frac{1}{1-r_m} \right)^\sigma \right). \quad (3)$$

For $r \in \left[r_m, \frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m \right]$, we have

$$\lim_{m \rightarrow +\infty} \frac{\exp \left(\tau \left(\log_{q-1} \left(1 - \frac{1}{m}\right) \left(\frac{1}{1-r}\right) \right)^\sigma \right)}{\exp \left(\tau \left(\log_{q-1} \frac{1}{1-r} \right)^\sigma \right)} = 1.$$

Then for any given $0 < \beta < \tau^* - \varepsilon$, there exists a positive integer m_2 such that for all $m \geq m_2$, and for all $r \in \left[r_m, \frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m \right]$, we have

$$\frac{\exp \left(\tau \left(\log_{q-1} \left(1 - \frac{1}{m}\right) \left(\frac{1}{1-r}\right) \right)^\sigma \right)}{\exp \left(\tau \left(\log_{q-1} \frac{1}{1-r} \right)^\sigma \right)} > \frac{\beta}{\tau^* - \varepsilon}. \quad (4)$$

By (3) and (4), for all $m \geq m_3 = \max \{m_1, m_2\}$ and for all

$$r \in \left[r_m, \frac{1}{m} + \left(1 - \frac{1}{m}\right) r_m \right],$$

we have

$$\begin{aligned}
\log_{p-2} T(r, f) &\geq \log_{p-2}^+ T(r_m, f) \\
&> (\tau^* - \varepsilon) \exp \left(\tau \left(\log_{q-1} \frac{1}{1-r_m} \right)^\sigma \right) \\
&\geq (\tau^* - \varepsilon) \exp \left(\tau \left(\log_{q-1} \left(1 - \frac{1}{m} \right) \left(\frac{1}{1-r} \right) \right)^\sigma \right) \\
&> \beta \exp \left(\tau \left(\log_{q-1} \frac{1}{1-r} \right)^\sigma \right).
\end{aligned}$$

Set

$$E = \bigcup_{m=m_3}^{+\infty} \left[r_m, \frac{1}{m} + \left(1 - \frac{1}{m} \right) r_m \right].$$

Then

$$\int_E \frac{dt}{1-t} = \sum_{m=m_3}^{+\infty} \int_{r_m}^{\frac{1}{m} + (1-\frac{1}{m})r_m} \frac{dt}{1-t} = \sum_{m=m_3}^{+\infty} \log \frac{m}{m-1} = +\infty.$$

□

Similarly, as in Lemma 2.4, we can get the following lemma.

Lemma 2.6. *Let $p \geq 2$ and $1 \leq q \leq p$ and f be a meromorphic function in \mathbb{D} with $0 < \mu_{[p,q]}(f) = \mu < \infty$, $0 < \underline{\tau}_{[p,q]}(f) = \underline{\tau} < \infty$ and $0 < \underline{\tau}_{[p,q]}^*(f) = \underline{\tau}^* < \infty$. Then, for any given $\varepsilon > 0$, there exists a set $E \subset [0, 1)$ that has an infinite logarithmic measure $\int_E \frac{dt}{1-t} = +\infty$ such that*

$$T(r, f) < \exp_{p-2} \left\{ (\underline{\tau}^* + \varepsilon) \exp \left(\underline{\tau} \left[\log_{q-1} \frac{1}{1-r} \right]^\mu \right) \right\}$$

holds for all $r \in E$.

Lemma 2.7 ([1]). *Let $p \geq q \geq 1$ be integers. If $A_0(z), \dots, A_{k-1}(z)$ are analytic functions of $[p, q]$ -order in the unit disc Δ , then every solution $f \not\equiv 0$ of (1) satisfies*

$$\rho_{[p+1,q]}(f) = \rho_{M,[p+1,q]}(f) \leq \max \{ \rho_{M,[p,q]}(A_j) : j = 0, 1, \dots, k-1 \}.$$

3 Proof of Theorem 1.15

Let $\mu_{[p,q]}(B) = \mu$ and let $f \not\equiv 0$ be a solution of the equation (2). We have

$$-B(z) = \frac{f''}{f} + A(z) \frac{f'}{f}. \quad (5)$$

Then,

$$m(r, B) \leq m(r, A) + m\left(r, \frac{f''}{f}\right) + m\left(r, \frac{f'}{f}\right) + O(1). \quad (6)$$

By Lemma 2.1 and as $A(z)$ and $B(z)$ are all analytic, then from (6) the following holds

$$T(r, B) \leq T(r, A) + O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right) \quad (7)$$

for all $r \notin F$, where $\int_F \frac{dt}{1-t} < \infty$. Now, from hypotheses of Theorem 1.15 we set $\mu_{[p,q]}(A) = \beta < \mu$. Then, by Lemma 2.3, for any given ε with $0 < 2\varepsilon < \mu - \beta$ and for all $r \in E$ with $\int_E \frac{dt}{1-t} = \infty$, we get

$$T(r, A) < \exp_p \left\{ (\beta + \varepsilon) \log_q \left(\frac{1}{1-r} \right) \right\} \quad (8)$$

and we have for $r \rightarrow 1^-$

$$T(r, B) > \exp_p \left\{ (\mu - \varepsilon) \log_q \left(\frac{1}{1-r} \right) \right\}. \quad (9)$$

By substituting (8) and (9) into (7), we obtain for $r \in E - F, r \rightarrow 1^-$

$$\begin{aligned} \exp_p \left\{ (\mu - \varepsilon) \log_q \left(\frac{1}{1-r} \right) \right\} < \exp_p \left\{ (\beta + \varepsilon) \log_q \left(\frac{1}{1-r} \right) \right\} \\ + O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right). \end{aligned} \quad (10)$$

By noting that $\mu - \varepsilon > \beta + \varepsilon$, it follows from (10) that

$$(1 - o(1)) \exp_p \left\{ (\mu - \varepsilon) \log_q \left(\frac{1}{1-r} \right) \right\} < O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right). \quad (11)$$

Hence, by (11) and since $\varepsilon > 0$ is arbitrary, we get

$$\sigma_{[p,q]}(f) = +\infty$$

and

$$\sigma_{[p+1,q]}(f) \geq \mu_{[p+1,q]}(f) \geq \mu = \mu_{[p,q]}(B).$$

4 Proof of Theorem 1.21

By Lemma 2.1 and Lemma 2.4, and as same reasoning as in the proof of Theorem 1.15, we get the result.

5 Proof of Theorem 1.26

Let $\sigma_{[p,q]}(B) = \sigma$ and let $f \not\equiv 0$ be a solution of the equation (2). Then f satisfies

$$-B(z) = \frac{f''}{f} + A(z) \frac{f'}{f} \quad (12)$$

By (12) and Lemma 2.1, we obtain

$$\begin{aligned} T(r, B) = m(r, B) &\leq m(r, A) + m\left(r, \frac{f''}{f}\right) + m\left(r, \frac{f'}{f}\right) + O(1) \\ &= T(r, A) + O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right) \end{aligned} \quad (13)$$

for all $r \in F$, where $F \subset [0, 1)$ has finite logarithmic measure. From the hypotheses of Theorem 1.26, there exist real constants β and β_1 such that $\tau_{[p,q]}^*(A) < \beta_1 < \beta < \tau_{[p,q]}^*(B) = \tau^*$. Then for $r \rightarrow 1^-$, we have

$$T(r, A) \leq \exp_{p-2} \left(\beta_1 \exp \left(\tau \left(\log_{q-1} \frac{1}{1-r} \right)^\sigma \right) \right). \quad (14)$$

By Lemma 2.5, there exists a subset $E \subset [0, 1)$ that has an infinite logarithmic measure such that for all $r \in E$, we have

$$T(r, B) > \exp_{p-2} \left(\beta \exp \left(\tau \left(\log_{q-1} \frac{1}{1-r} \right)^\sigma \right) \right). \quad (15)$$

From (13)-(15) we obtain for $r \in E - F$ that

$$\begin{aligned} \exp_{p-2} \left(\beta \exp \left(\tau \left(\log_{q-1} \frac{1}{1-r} \right)^\sigma \right) \right) &< \exp_{p-2} \left(\beta_1 \exp \left(\tau \left(\log_{q-1} \frac{1}{1-r} \right)^\sigma \right) \right) \\ &+ O\left(\log^+ T(r, f) + \log\left(\frac{1}{1-r}\right)\right). \end{aligned} \quad (16)$$

By (16), we obtain

$$\sigma_{[p,q]}(f) = +\infty$$

and

$$\sigma_{[p+1,q]}(f) \geq \sigma = \sigma_{[p,q]}(B).$$

On the other hand, by Lemma 2.7, we have

$$\rho_{[p+1,q]}(f) \leq \max \{ \rho_{M,[p,q]}(A), \rho_{M,[p,q]}(B) \}.$$

It yields

$$\sigma_{[p,q]}(B) \leq \rho_{[p+1,q]}(f) \leq \max \{ \rho_{M,[p,q]}(A), \rho_{M,[p,q]}(B) \}.$$

If $p > q$, then we have

$$\max \{ \rho_{M,[p,q]}(A), \rho_{M,[p,q]}(B) \} = \sigma_{[p,q]}(B).$$

Therefore, we deduce that

$$\rho_{[p+1,q]}(f) = \sigma_{[p,q]}(B).$$

6 Proof of Theorem 1.28

By Lemma 2.1 and Lemma 2.6, and as same reasoning as in proof of Theorem 1.26, we get the result.

7 Conclusion

Throughout this article, we have shown firstly that the similarity between the cases of usual orders and lower orders is also satisfied in the unit disc \mathbb{D} , and we have generalized the previous results to general $[p, q]$ -growth. Defining new type of growth in the unit disc \mathbb{D} is discussed and is applied to complex differential equations to solve some problems related to growth of solutions.

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