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Remark on the Irregularity of Graphs

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Abstract: Let $G = (V,E)$, $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$, be a simple connected graph with the vertex degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, $d_i = d(v_i)$. The zeroth–order general Randić index, ${}^0R_\alpha(G)$, of a connected graph G, is defined as ${}^0R_\alpha(G) = \sum_{i=1}^{n} d_i^\alpha$. A linear combination of ${}^0R_\alpha(G)$ of the form $irr^{(\alpha)}(G) = {}^0R_{\alpha+1}(G) - \frac{2m}{n}{}^0R_\alpha(G)$, $\alpha \geq 0$, can be considered as an irregularity measure of a graph since $irr^{(\alpha)}(G) = 0$ if and only if G is a regular graph, and $irr^{(\alpha)}(G) > 0$ otherwise. In this paper we consider a linear combination $irr^{(\alpha)}(G) - \frac{2m}{n} irr^{(\alpha-1)}(G)$, for $\alpha \geq 1$, which can be also considered as irregularity measure of graph, and determine its bounds.

Keywords: Topological indices, irregularity (of a graph).

1 Introduction

Let $G = (V,E)$, $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$, be a simple connected graph with the vertex degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, $d_i = d(v_i)$.

A topological index, or graph invariant, for a graph is a numerical quantity which is invariant under isomorphism of the graph. The study of the mathematical aspects of the degree-based graph invariants (also known as topological indices) is considered to be one of the very active research areas within the field of chemical graph theory.

The first Zagreb index [1] is a vertex–degree based graph invariant defined as [2]

$$M_1(G) = \sum_{i=1}^{n} d_i^2 .$$

The first Zagreb index is the oldest and most extensively studied graph–based molecular structure descriptor. Details about its applications and mathematical properties can be found in surveys [3–7] and in the references cited therein.

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Various generalizations of the first Zagreb index have been proposed. In [8] a so called zeroth–order general Randić index was introduced. It is defined as

$$^0R_\alpha(G) = \sum_{i=1}^{n} d_i^\alpha ,$$

where α is an arbitrary real number. This index is also met in the literature under the names first general Zagreb index [9], or variable first Zagreb index [10]. For some particular values of α the following indices/values are obtained

- The modified first Zagreb index [7] is obtained for $\alpha = -2$, that is ${}^mM_1(G) = {}^0R_{-2}(G)$;
- The inverse degree index [11] is obtained for $\alpha = -1$, that is $ID(G) = {}^0R_{-1}(G)$;
- For $\alpha = 0$ we have ${}^0R_0(G) = n$;
- For $\alpha = 1$, we have ${}^0R_1(G) = \sum_{i=1}^{n} d_i = 2m$;
- The first Zagreb index is obtained for $\alpha = 2$, $M_1(G) = {}^0R_2(G)$
- The forgotten topological index [12] is obtained for $\alpha = 3$, $F(G) = {}^0R_3(G)$.

A graph G is called regular if all its vertices have the same degree. Any mapping that associates a real number $IM(G)$ to a graph G, satisfying the condition $IM(G) = 0$ if and only if G is regular, and $IM(G) > 0$ otherwise, can be used as an irregularity measure. On various irregularity measures the reader can refer to [13–21].

2 Preliminaries

In [14] it was proven that for any real $\alpha \geq 0$ holds

$$^0R_{\alpha+1}(G) \geq \frac{2m}{n}\, {}^0R_\alpha(G) , \tag{2.1}$$

with equality if and only if $\alpha = 0$, or G is regular. When $\alpha < 0$ the sense of inequality (2.1) reverses, that is

$$^0R_\alpha(G) \geq \frac{n}{2m}\, {}^0R_{\alpha+1}(G) . \tag{2.2}$$

From the inequality (2.1) for $\alpha > 0$, a number of irregularity measures can be derived:

$$irr^{(\alpha)}(G) = {}^0R_{\alpha+1}(G) - \frac{2m}{n}\, {}^0R_\alpha(G) . \tag{2.3}$$

Thus, for $\alpha = 1$, we obtain the well known Edwards irregularity measure [13,20,23]:

$$irr^{(1)}(G) = M_1(G) - \frac{4m^2}{n} . \tag{2.4}$$

This irregularity measure is closely related to the irregularity measure introduced by Bell [14], and defined as

$$irr_B(G) = \frac{1}{n}\sum_{i=1}^{n}\left(d_i - \frac{2m}{n}\right)^2 .$$

Namely, the following is valid [13]:

$$irr^{(1)}(G) = n \cdot irr_B(G) .$$

In [14] the following inequality was proven

$$\frac{1}{2}(\Delta-\delta)^2 \le irr^{(1)}(G) \le n\alpha(n)(\Delta-\delta)^2 , \tag{2.5}$$

where

$$\alpha(n) = \frac{1}{4}\left(1 - \frac{1+(-1)^{n+1}}{2n^2}\right) .$$

From the inequality (2.2) for $\alpha < 0$, one can derive the following family of irregularity measures

$$irr^{(\alpha)}(G) = {}^0R_\alpha(G) - \frac{n}{2m}\,{}^0R_{\alpha+1}(G) . \tag{2.6}$$

Here we are interested for the irregularity measure obtained from (2.6) for the case $\alpha = -1$, that is

$$irr^{(-1)}(G) = ID(G) - \frac{n^2}{2m} . \tag{2.7}$$

In [25] the following inequalities were proven

$$\frac{1}{\Delta+\delta}\left(\sqrt{\frac{\Delta}{\delta}} - \sqrt{\frac{\delta}{\Delta}}\right)^2 \le irr^{(-1)}(G) \le \frac{n(n-1)}{4m}\left(\sqrt{\frac{\Delta}{\delta}} - \sqrt{\frac{\delta}{\Delta}}\right)^2 . \tag{2.8}$$

In [26] it was proven that for any real $\alpha \ge 0$ holds

$${}^0R_{\alpha+1}(G) \ge \frac{(2m)^{\alpha+1}}{n^\alpha} . \tag{2.9}$$

It is not difficult to observe that the above inequality also holds when $\alpha \le -1$, and that when $-1 \le \alpha \le 0$ the opposite inequality is valid. Equality in (2.9) holds if and only if either $\alpha = -1$, or $\alpha = 0$, or G is a regular graph.

For $\alpha \le -1$ or $\alpha \ge 0$, from the inequality (2.9) one can derive a family of irregularity measures of the form

$$irr_\alpha(G) = {}^0R_{\alpha+1}(G) - \frac{(2m)^{\alpha+1}}{n^\alpha} . \tag{2.10}$$

It is not difficult to observe that for $\alpha = 1$ (2.10) coincides with (2.4), and for $\alpha = -2$ it coincides with (2.7), that is that the following is valid

$$irr^{(1)}(G) = irr_1(G) \quad \text{and} \quad irr^{(-1)}(G) = irr_{-2}(G) .$$

Particulary interesting for us are irregularity measures obtained from (2.10) for $\alpha=-3$ and $\alpha=2$, that is

$$irr_{-3}(G)={}^{m}M_1(G)-\frac{n^3}{4m^2} \qquad \text{and} \qquad irr_2(G)=F(G)-\frac{8m^3}{n^2}\,. \tag{2.11}$$

In this paper we consider bounds of the expression

$$irr^{(\alpha)}(G)-\frac{2m}{n}irr^{(\alpha-1)}(G)\,, \tag{2.12}$$

where α is an arbitrary real number. When $\alpha\geq 1$ the expression (2.12) can be considered as an irregularity measure. New bounds for topological indices $M_1(G)$ and $F(G)$ are obtained as special cases.

3 Main results

First we recall one inequality for real number sequences that will be frequently used later in this paper.

Lemma 3.1. [27] *Let $p=(p_i)$, $i=1,2,\dots,n$, be a sequence of non negative real numbers and $a=(a_i)$, $i=1,2,\dots,n$, a sequence of positive real numbers. Then, for any real r, $r\leq 0$ or $r\geq 1$, we have that*

$$\left(\sum_{i=1}^{n}p_i\right)^{r-1}\sum_{i=1}^{n}p_ia_i^r\geq\left(\sum_{i=1}^{n}p_ia_i\right)^{r}. \tag{3.1}$$

When $0\leq r\leq 1$, the opposite inequality is valid. Equality holds if and only if either $r=0$, or $r=1$, or $a_1=a_2=\cdots=a_n$, or $p_1=\cdots=p_t=0$ and $a_{t+1}=\cdots=a_n$, or $a_1=\cdots=a_t$ and $p_{t+1}=\cdots=p_n=0$, for some t, $1\leq t\leq n-1$.

More on the above inequality can be found in [28,29].

Theorem 3.1. *Let G be a connected irregular graph with $n\geq 3$ vertices and m edges. Then, for any real α, $\alpha\leq 0$ or $\alpha\geq 1$, we have that*

$$irr^{(\alpha-1)}(G)-\frac{2m}{n}irr^{(\alpha-2)}(G)\geq\frac{4\left(\frac{m}{n}\right)^{\alpha+1}irr_{-2}(G)^{\alpha}}{\left(\frac{m}{n}irr_{-3}(G)-irr_{-2}(G)\right)^{\alpha-1}}\,. \tag{3.2}$$

When $0\leq\alpha\leq 1$ the opposite inequality is valid. Equality holds if and only if $\alpha=0$ or $\alpha=1$.

Proof. For $r=\alpha$, $\alpha\leq 0$ or $\alpha\geq 1$, $p_i=\frac{\left(d_i-\frac{2m}{n}\right)^2}{d_i^2}$, $a_i=d_i$, $i=1,2,\dots,n$, the inequality (3.1) becomes

$$\left(\sum_{i=1}^{n}\frac{\left(d_i-\frac{2m}{n}\right)^2}{d_i^2}\right)^{\alpha-1}\sum_{i=1}^{n}\left(d_i-\frac{2m}{n}\right)^2d_i^{\alpha-2}\geq\left(\sum_{i=1}^{n}\frac{\left(d_i-\frac{2m}{n}\right)^2}{d_i}\right)^{\alpha}. \tag{3.3}$$

On the other hand, the following identities are valid

$$\sum_{i=1}^{n} \frac{\left(d_i - \frac{2m}{n}\right)^2}{d_i^2} = \sum_{i=1}^{n} \left(1 - \frac{4m}{n}\frac{1}{d_i} + \frac{4m^2}{n^2}\frac{1}{d_i^2}\right) =$$

$$= n - \frac{4m}{n} ID(G) + \frac{4m^2}{n^2}\, {}^{m}M_1(G) =$$

$$= 2n - \frac{4m}{n} ID(G) + \frac{4m^2}{n^2}\, {}^{m}M_1(G) - n =$$

$$= \frac{4m^2}{n^2}\left({}^{m}M_1(G) - \frac{n^3}{4m^2}\right) - \frac{4m}{n}\left(ID(G) - \frac{n^2}{2m}\right).$$

From the above identity and inequalities (2.7) and (2.11), we obtain

$$\sum_{i=1}^{n} \frac{\left(d_i - \frac{2m}{n}\right)^2}{d_i^2} = \frac{4m}{n}\left(\frac{m}{n} irr_{-3}(G) - irr_{-2}(G)\right). \tag{3.4}$$

Since

$$\sum_{i=1}^{n} \left(d_i - \frac{2m}{n}\right)^2 d_i^{\alpha-2} = \sum_{i=1}^{n} \left(d_i^{\alpha} - \frac{4m}{n} d_i^{\alpha-1} + \frac{4m^2}{n^2} d_i^{\alpha-2}\right) =$$

$$= {}^{0}R_{\alpha}(G) - \frac{4m}{n}\, {}^{0}R_{\alpha-1}(G) + \frac{4m^2}{n^2}\, {}^{0}R_{\alpha-2}(G) =$$

$$= {}^{0}R_{\alpha}(G) - \frac{2m}{n}\, {}^{0}R_{\alpha-1}(G) - \frac{2m}{n}\left({}^{0}R_{\alpha-1}(G) - \frac{2m}{n}\, {}^{0}R_{\alpha-2}(G)\right) =$$

$$= irr^{(\alpha-1)}(G) - \frac{2m}{n} irr^{(\alpha-2)}(G), \tag{3.5}$$

and

$$\sum_{i=1}^{n} \frac{\left(d_i - \frac{2m}{n}\right)^2}{d_i} = \sum_{i=1}^{n} \left(d_i - \frac{4m}{n} + \frac{4m^2}{n^2}\frac{1}{d_i}\right) =$$

$$= 2m - 4m + \frac{4m^2}{n^2} ID(G) = \frac{4m^2}{n^2}\left(ID(G) - \frac{n^2}{2m}\right) = \frac{4m^2}{n^2} irr_{-2}(G), \tag{3.6}$$

from identities (3.4), (3.5), (3.6) and inequality (3.3) we obtain

$$\left(\frac{4m}{n}\right)^{\alpha-1}\left(\frac{m}{n} irr_{-3}(G) - irr_{-2}(G)\right)^{\alpha-1}\left(irr^{(\alpha-1)}(G) - \frac{2m}{n} irr^{(\alpha-2)}(G)\right) \geq$$

$$\geq \left(\frac{4m^2}{n^2}\right)^{\alpha} irr_{-2}(G)^{\alpha}.$$

Since G is irregular, we have that $irr_{-2}(G) > 0$ and $\frac{m}{n} irr_{-3}(G) - irr_{-2}(G) > 0$, from the above inequality follows (3.2).

The case when $0 \leq \alpha \leq 1$ can be proved similarly. Since G is irregular, equality in (3.3), and consequently in (3.2), holds if and only if $\alpha = 0$ or $\alpha = 1$. □

Corollary 3.1. *Let G be a connected irregular graph with $n \geq 3$ vertices and m edges. Then we have*

$$M_1(G) \geq \frac{4m^2}{n} + \frac{4\left(\frac{m}{n}\right)^3 irr_{-2}(G)^2}{\frac{m}{n} irr_{-3}(G) - irr_{-2}(G)}. \tag{3.7}$$

Proof. For $\alpha = 2$ the inequality (3.2) becomes

$$irr^{(1)}(G) - \frac{2m}{n} irr^{(0)}(G) \geq \frac{4\left(\frac{m}{n}\right)^3 irr_{-2}(G)^2}{\frac{m}{n} irr_{-3}(G) - irr_{-2}(G)}. \tag{3.8}$$

Since

$$irr^{(0)}(G) = {}^0R_1(G) - \frac{2m}{n}\,{}^0R_0(G) = 2m - 2m = 0,$$

and having in mind identity (2.4) and inequality (3.8), we arrive at (3.7). □

Remark 3.1. *The inequality* (3.7) *is stronger than*

$$M_1(G) \geq \frac{4m^2}{n} + \frac{4\left(\frac{m}{n}\right)^2 irr_{-2}(G)^2}{irr_{-3}(G)},$$

which was proven in [30].

Corollary 3.2. *Let G be a connected irregular graph with $n \geq 3$ vertices and m edges. Then we have*

$$F(G) \geq \frac{4m}{n} M_1(G) - \frac{8m^3}{n^2} + \frac{4\left(\frac{m}{n}\right)^4 irr_{-2}(G)^3}{\left(\frac{m}{n} irr_{-3}(G) - irr_{-2}(G)\right)^2}.$$

Corollary 3.3. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$F(G) - \frac{8m^3}{n^2} \geq \frac{4m}{n}\left(M_1(G) - \frac{4m^2}{n}\right),$$

$$F(G) \geq \frac{2m}{n} M_1(G), \tag{3.9}$$

$$F(G) \geq \frac{8m^3}{n^2}. \tag{3.10}$$

Equalities hold if and only if G is regular.

Remark 3.2. *The inequality* (3.9) *was proven in* [31], *whereas* (3.10) *in* [9] (*see also* [26]).

Corollary 3.4. *Let U, $U \ncong C_n$, be a connected unicyclic graph with $n \geq 4$ vertices. Then*

$$M_1(U) \geq 4n + \frac{4 irr_{-2}(U)^2}{irr_{-3}(U) - irr_{-2}(U)}, \tag{3.11}$$

and

$$F(U) \geq 4M_1(U) - 8n + \frac{4 irr_{-2}(U)^3}{\left(irr_{-3}(U) - irr_{-2}(U)\right)^2}. \tag{3.12}$$

Remark 3.3. *When U, $U \ncong C_n$, is connected unicyclic graph with $n \geq 4$, the inequality* (3.11) *is stronger than*

$$M_1(U) \geq 4n\,,$$

which was proven in [32]*, whereas the inequality* (3.12) *is stronger than*

$$F(U) \geq 4M_1(U) - 8n\,.$$

The proof of the next theorem is analogous to that of Theorem 3.1, hence omitted.

Theorem 3.2. *Let G be a connected irregular graph with $n \geq 3$ vertices and m edges. Then, for any real α, $\alpha \leq 0$ or $\alpha \geq 1$, we have that*

$$irr^{(\alpha)}(G) - \frac{2m}{n} irr^{(\alpha-1)}(G) \geq \frac{n^{2\alpha-2} irr_1(G)^{\alpha}}{(2m)^{2\alpha-2} irr_{-2}(G)^{\alpha-1}}\,. \tag{3.13}$$

When $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality holds if and only if $\alpha = 0$ or $\alpha = 1$.

Corollary 3.5. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then for any real $\alpha > 1$, holds*

$$irr^{(\alpha)}(G) - \frac{2m}{n} irr^{(\alpha-1)}(G) \geq \frac{n^{\alpha-1}(\Delta-\delta)^2(\Delta\delta)^{\alpha-1}}{2^{\alpha} m^{\alpha-1}(n-1)^{\alpha-1}}\,.$$

Equality holds if and only if G is regular.

Proof. The required result immediately follows from (3.13), left–hand side of (2.5) and right–hand side of (2.8). □

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References

[1] A. T. Balaban I. Motoc, D. Bonchev, D. Mekenyan, *Topological indices for structure–activity correlations*, Topics Curr. Chem. 111 (1983) 21–55.

[2] I. Gutman, N. Trinajstić, *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. 17 (1972) 535–538.

[3] A. Ali, I. Gutman, E. Milovanović, I. Milovanović, *Sum of powers of the degrees of graphs: extremal results and bounds*, MATCH Commun. Math. Comput. Chem. 80 (2018) 5–84.

[4] B. Borovićanin, K. C. Das, B. Furtula, I. Gutman, *Bounds for Zagreb indices*, MATCH Commun. Math. Comput. Chem. 78 (2017) 17–100.

[5] B. BOROVIĆANIN, K. C. DAS, B. FURTULA, I. GUTMAN, *Zagreb indices: Bounds and extremal graphs*, In: Bounds in Chemical Graph Theory – Basics, I. Gutman, B. Furtula, K. C. Das, E. Milovanović, I. Milovanović, Eds., Univ. Kargujevac, Kragujevac, 2017, pp. 67–153.

[6] I. GUTMAN, K. C. DAS, *The first Zagreb index 30 years after*, MATCH Commun. Math. Comput. Chem. 50 (2004) 83–92.

[7] S. NIKOLIĆ, G. KOVAČEVIĆ, A. MILIČEVIĆ, N. TRINAJSTIĆ, *The Zagreb indices 30 years after*, Croat. Chem. Acta 70 (2003) 113–124.

[8] Y. HO, X. LI, Y. SHI, T. XU, I. GUTMAN, *On molecular graphs with smallest and greatest zeroth–order general Randić index*, MATCH Commun. Math. Comput. Chem. 54 (2005) 425–434.

[9] X. LI, J. ZHANG, *A unified approach to the extremal trees for different indices*, MATCH Commun. Math. Comput. Chem. 54 (2005) 195–208.

[10] A. MILIĆEVIĆ, S. NIKOLIĆ, *On variable Zagreb indices*, Croat. Chem. Acta 77 (2004) 97–101.

[11] S. FAJTLOWICZ, *On conjectures on Graffiti-II*, Congr. Numer. 60 (1987) 187–197.

[12] B. FURTULA, I. GUTMAN, *A forgotten topological index*, J. Math. Chem. 53 (2015) 1184–1190.

[13] I. GUTMAN, *Topological indices and irregularity measures*, Bull. Inter. Math. Virt. Inst. 8 (2018) 469–475.

[14] F. K. BELL, *A note on the irregularity of graphs*, Linear Algebra Appl. 16 (1992) 45–54.

[15] A. HAMZEH, T. RETI, *An analogue of Zagreb index inequality obtained from graph irregularity measures*, MATCH Commun. Math. Comput. Chem. 72 (2014) 669–683.

[16] V. NIKIFOROV, *Eigenvalues and degree deviation in graphs*, Linear Algebra Appl. 414 (2006) 347–360.

[17] I.Ž. MILOVANOVIĆ, E. I. MILOVANOVIĆ, V. ĆIRIĆ, N. JOVANOVIĆ, *On some irregularity measures of graphs*, Sci. Publ. State Univ. Novi Pazar Ser. A: Appl. Math. Inform. Mech. 8(1) (2016) 21–34.

[18] A. ALI, E. MILOVANOVIĆ, M. MATEJIĆ, I. MILOVANOVIĆ, *On the upper bounds for the degree–deviation of graphs*, J. Appl. Math. Comput. 62 (2020) 179–187.

[19] I. GUTMAN, B. FURTULA, C. ELPHICK, *Three new/old vertex–degree–based topological indices*, MATCH Commun. Math. Comput. Chem. 72 (2014) 617–632.

[20] T. RETI, A. ALI, *On the variance–type graph irregularity measures*, Commun. Comb. Optim. 5(2) (2020) 169–178.

[21] T. RETI, I. MILOVANOVIĆ, E. MILOVANOVIĆ, M. MATEJIĆ, *On graph irregularity indices with particular regard to degree deviation*, Filomat 35 (11) (2021) 3689–3701.

[22] S. M. CIOABA, *Sums of powers of the degree of a graph*, Discr. Math. 306 (2006) 1959–1964.

[23] C. S. EDWARDS, *The largest vertex degree sum for a triangle in a graph*, Bull. London Math. Soc. 9 (1977) 203–208.

[24] E. I. MILOVANOVIĆ, I. Ž. MILOVANOVIĆ, *Sharp bounds for the first Zagreb index and first Zagreb coindex*, Miskolc Math. Notes 16(2) (2015) 1017–1024.

[25] I. GUTMAN, K. CH. DAS, B. FURTULA, E. MILOVANOVIĆ, I. MILOVANOVIĆ, *Generalizations of Szökefalvi Nagy and Chebyshev inequalities with applications in spectral graph theory*, Appl. Math. Comput. 313 (2017) 235–244.

[26] A. ILIĆ, D. STEVANOVIĆ, *On comparing Zagreb indices*, MATCH Commun. Math. Comput. Chem. 62 (2009) 681–687.

[27] J. L. W. V. JENSEN, *Sur les fonctions convexes of les integralites entre les values moyennes*, Acta Math. 30 (1906) 175–193.

[28] D. S. MITRINOVIĆ, P. M. VASIĆ, *The centroid method in inequalities*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 498–541 (1975) 3–16.

[29] D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK, Inequalities involving functions and their integrals and derivatives, Kluwer Academic Publishers, Dordrecht–Boston–London, 1991.

[30] S. D. STANKOV, M. M. MATEJIĆ, I. Ž. MILOVANOVIĆ, E. I. MILOVANOVIĆ, Ş. B. BOZKURT ALTINDAĞ, *Some new bounds on the Zagreb index*, Electron. J. Math. 1 (2021) 101–107.

[31] B. FURTULA, I. GUTMAN, Ž. KOVIJANIĆ VUKIČEVIĆ, G. LEKISHVILI, G. POPIVODA, *On an old/new degree–based topological index*, Bull. Acad. Serbe Sci. Arts (Cl. Sci Math. Natur) 40 (2015) 19–31.

[32] S. ZHANG, H. ZHANG, *Unicyclic graphs with the first three smallest and largest general Zagreb index*, MATCH Commun. Math. Comput. Chem. 55 (2006) 427–438.