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Bounds for the reverse (delta) first Zagreb indices

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Abstract: Let $G = (V, E)$ be a simple connected graph with $n \geq 2$ vertices and m edges. In the literature four vertex–degree–based topological indices are associated to topological index, known as the first Zagreb index, based on the modification of a sequence of degrees. In this paper we study relationships between these topological indices and establish some new bounds.
Keywords: Graphs, topological indices and coindices, degree-based invariants.

1 Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \dots, v_n\}$, be a simple connected graph with n vertices, m edges with vertex–degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$, $d_i = d(v_i)$. If vertices v_i and v_j are adjacent in G, we write $i \sim j$, otherwise we write $i \nsim j$.

In graph theory, a graph invariant is property of the graph that is preserved by isomorphisms. The graph invariants that assume only numerical values are usually referred to as topological indices in chemical graph theory. Hundreds of various topological indices have been introduced in mathematical chemistry literature in order to describe physical and chemical properties of molecules (see for example [19–21]). Many of them are defined as simple functions of the degree sequence of (molecular) graph. Most of the degree–based topological indices are viewed as the contributions of pairs of adjacent vertices. These type of indices are also known as the bond incident degree (BID in short) indices [22]. Various mathematical properties of topological indices have been investigated, as well.

The first Zagreb index, $M_1(G)$, is the oldest and most thoroughly examined vertex–degree–based topological index. It is defined as sum of squares of vertex degrees [12], that is

$$M_1(G) = \sum_{i=1}^{n} d_i^2. \tag{1.1}$$

It is known that $M_1(G)$ can be also represented as [18]

$$M_1(G) = \sum_{i \sim j} (d_i + d_j) \tag{1.2}$$

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The first Zagreb index became one of the most popular and most extensively studied graph-based molecular structure descriptors. More on its mathematical properties and chemical applications can be found in [3,4,13,14,18] and in the references cited therein.

The inverse degree index of graph G without isolated vertices is defined as the sum of reciprocal of vertex degrees of the graph [11]

$$ID(G) = \sum_{i=1}^{n} \frac{1}{d_i},$$

The inverse degree first attracted attention through conjectures of the computer program Graffiti [11].

A new vertex–degree–like sequence was introduced in [16] as $\Delta - \delta + 1 = s_1 \geq s_2 \geq \cdots \geq s_n = 1$, where

$$s_i = s_i(G) = d_i - \delta + 1, \tag{1.3}$$

for $i = 1, 2, \dots, n$. Then, by analogy with (1.1), the delta first Zagreb index was put forward as

$$DM_1(G) = \sum_{i=1}^{n} s_i^2. \tag{1.4}$$

Another vertex–degree–like sequence, $1 = c_1 \leq c_2 \leq \cdots \leq c_n = \Delta - \delta + 1$, where

$$c_i = c_i(G) = \Delta - d_i + 1, \tag{1.5}$$

for $i = 1, 2, \dots, n$ was introduced in [8] (see also [9, 10]). By analogy with (1.1) and (1.2), new graph invariants named as reverse first Zagreb alpha index, $RM_1^{\alpha}(G)$, and reverse first Zagreb beta index, $RM_1^{\beta}(G)$, were defined as

$$RM_1^{\alpha}(G) = \sum_{i=1}^{n} c_i^2 \quad \text{and} \quad RM_1^{\beta}(G) = \sum_{i \sim j} (c_i + c_j). \tag{1.6}$$

In the light of the above, delta first Zagreb index defined by (1.4) could be called delta first Zagreb alpha index and denoted by

$$DM_1^{\alpha}(G) = DM_1(G),$$

and delta first Zagreb beta index, $DM_1^{\beta}(G)$, could be defined as

$$DM_1^{\beta}(G) = \sum_{i \sim j} (s_i + s_j). \tag{1.7}$$

One can easily see that the following identities hold (see, for example, [1, 7])

$$DM_1^{\beta}(G) = \sum_{i \sim j} (s_i + s_j) = \sum_{i=1}^{n} d_i s_i \quad \text{and} \quad RM_1^{\beta}(G) = \sum_{i \sim j} (c_i + c_j) = \sum_{i=1}^{n} d_i c_i. \tag{1.8}$$

The concept of coindices was introduced in [6](see also [2]). In this case the sum runs over the edges of the complement of G. In a view of (1.2), the corresponding first Zagreb coindex of G is defined as

$$\overline{M}_1(G) = \sum_{i \nsim j} (d_i + d_j).$$

Analogously, delta first Zagreb beta coindex, $\overline{DM}_1^\beta(G)$, and reverse first Zagreb beta coindex, $\overline{RM}_1^\beta(G)$, are defined as

$$\overline{DM}_1^\beta(G) = \sum_{i \nsim j} (s_i + s_j) \quad \text{and} \quad \overline{RM}_1^\beta(G) = \sum_{i \nsim j} (c_i + c_j) \,. \tag{1.9}$$

In [5] a relationship between the first Zagreb index and delta (reverse) first Zagreb index was considered. In the present paper we establish bounds for delta and reverse first Zagreb indices in terms of $ID(G)$ and some basic graph parameters.

2 Preliminaries

At the beginning we state one analytical inequality for real number sequences which will be used in the rest of the paper.

Lemma 2.1. [17] *Let $p = (p_i)$, $i = 1, 2, \dots, n$, be a sequence of nonnegative real numbers and $a = (a_i)$, $i = 1, 2, \dots, n$, sequence of positive real numbers. Then, for any r, $r \le 0$ or $r \ge 1$, holds*

$$\left(\sum_{i=1}^n p_i\right)^{r-1} \sum_{i=1}^n p_i a_i^r \ge \left(\sum_{i=1}^n p_i a_i\right)^r . \tag{2.1}$$

When $0 \le r \le 1$ the opposite inequality is valid. Equality holds if and only if either $r = 0$, or $r = 1$, or $a_1 = a_2 = \cdots = a_n$, or $p_1 = p_2 = \cdots = p_t = 0$ and $a_{t+1} = \cdots = a_n$, or $p_{t+1} = \cdots = p_n = 0$ and $a_1 = a_2 = \cdots = a_t$, for some t, $1 \le t \le n-1$.

The inequality (2.1) is known in the literature as Jensen's inequality. Original form of this inequality is proven in [15].

3 Main results

In the next theorem we determine a lower bound for $RM_1^\beta(G)$ in terms of topological index $ID(G)$ and parameters n, m and Δ.

Theorem 3.1. *If G is a d-regular graph with $n \ge 2$ vertices, then*

$$RM_1^\beta(G) = nd.$$

If G is connected, non–regular graph with $n \ge 3$ vertices and m edges, then

$$RM_1^\beta(G) \ge 2m + \frac{(n\Delta - 2m)^2}{\Delta ID(G) - n}. \tag{3.1}$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \le t \le n-1$.

Proof. For $r = 2$, $p_i = \frac{c_i - 1}{d_i}$, $a_i = d_i$, $i = 1, 2, \dots, n$, the inequality (2.1) becomes

$$\sum_{i=1}^n \frac{c_i - 1}{d_i} \sum_{i=1}^n (c_i - 1) d_i \ge \left(\sum_{i=1}^n (c_i - 1)\right)^2 . \tag{3.2}$$

Since

$$\sum_{i=1}^{n}\frac{c_i-1}{d_i}=\sum_{i=1}^{n}\frac{\Delta-d_i}{d_i}=\Delta ID(G)-n,$$
$$\sum_{i=1}^{n}(c_i-1)d_i=RM_1^{\beta}(G)-2m,$$
$$\sum_{i=1}^{n}(c_i-1)=\sum_{i=1}^{n}(\Delta-d_i)=n\Delta-2m,$$

from (3.2) it follows

$$(\Delta ID(G)-n)(RM_1^{\beta}(G)-2m)\geq(n\Delta-2m)^2. \tag{3.3}$$

If G is not regular graph, then $\Delta ID(G)-n>0$, therefore according to (3.3) we obtain (3.1).

If G is not regular graph, then equality in (3.2) holds if and only if $c_1=\cdots=c_t=1$ and $d_{t+1}=\cdots=d_n=\delta$, for some t, $1\leq t\leq n-1$. Therefore equality in (3.1) holds if and only if $\Delta=d_1=\cdots=d_t>d_{t+1}=\cdots=d_n=\delta$, for some t, $1\leq t\leq n-1$. □

Corollary 3.1. *Let G be a connected graph with $n\geq 2$ vertices and m edges. Then*

$$RM_1^{\beta}(G)\geq n\Delta\delta-2m(\delta-1). \tag{3.4}$$

Equality holds if and only if $\Delta=d_1=\cdots=d_t\geq d_{t+1}=\cdots=d_n=\delta$, for some t, $1\leq t\leq n-1$.

Proof. For every $i=1,2,\ldots,n$, it holds

$$(d_i-\delta)(\Delta-d_i)\geq 0, \tag{3.5}$$

that is

$$d_i+\frac{\delta\Delta}{d_i}\leq\Delta+\delta.$$

After summation the above inequality over i, $i=1,2,\ldots,n$, we get

$$\delta\Delta ID(G)\leq n(\Delta+\delta)-2m,$$

i.e.

$$\Delta ID(G)-n\leq\frac{n\Delta-2m}{\delta}.$$

From the above and (3.1) we arrive at (3.4).

Equality in (3.5) holds if and only if $d_i\in\{\Delta,\delta\}$ for every i, $i=1,2,\ldots,n$. If G is a regular graph, then equality in (3.4) is attained. Finally, equality in (3.4) holds if and only if $\Delta=d_1=\cdots=d_t\geq d_{t+1}=\cdots=d_n=\delta$, for some t, $1\leq t\leq n-1$. □

Corollary 3.2. *Let G be a connected graph with $n\geq 2$ vertices with the property $\delta=1$. Then*

$$RM_1^{\beta}(G)\geq n\Delta.$$

Equality holds if and only if $\Delta=d_1=\cdots=d_t\geq d_{t+1}=\cdots=d_n=1$, for some t, $1\leq t\leq n-1$.

Corollary 3.3. *Let G be a connected, non–regular graph with $n \geq 3$ vertices and m edges. Then*

$$RM_1^{\alpha}(G) \leq n(\Delta+1)^2 - 2(\Delta+2)m - \frac{(n\Delta-2m)^2}{\Delta ID(G)-n}. \tag{3.6}$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \leq t \leq n-1$.

Proof. From (1.6) and (1.8) we have that

$$\begin{aligned} RM_1^{\alpha}(G) + RM_1^{\beta}(G) &= \sum_{i=1}^{n} c_i^2 + \sum_{i\sim j}(c_i+c_j) = \sum_{i=1}^{n} c_i^2 + \sum_{i=1}^{n} d_i c_i = \sum_{i=1}^{n}(c_i+d_i)c_i \\ &= (\Delta+1)\sum_{i=1}^{n} c_i = (\Delta+1)(n(\Delta+1)-2m), \end{aligned}$$

i.e.

$$RM_1^{\alpha}(G) = (\Delta+1)(n(\Delta+1)-2m) - RM_1^{\beta}(G). \tag{3.7}$$

From the above and (3.1) we arrive at (3.6). □

Corollary 3.4. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$RM_1^{\alpha}(G) \leq n((\Delta+1)^2 - \Delta\delta) - 2m(\Delta-\delta+2). \tag{3.8}$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \leq t \leq n-1$.

Corollary 3.5. *Let G be a connected, non–regular graph with $n \geq 3$ vertices and m edges. Then*

$$\overline{RM}_1^{\beta}(G) \leq n((n-1)(\Delta+1)-2m) - \frac{(n\Delta-2m)^2}{\Delta ID(G)-n}. \tag{3.9}$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \leq t \leq n-1$.

Proof. From (1.8) and (1.9) we have that

$$\begin{aligned} \overline{RM}_1^{\beta}(G) + RM_1^{\beta}(G) &= \sum_{i=1}^{n}(n-1-d_i)c_i + \sum_{i=1}^{n} d_i c_i = (n-1)\sum_{i=1}^{n} c_i \\ &= (n-1)(n(\Delta+1)-2m). \end{aligned} \tag{3.10}$$

From the above and (3.1) we get (3.9). □

Corollary 3.6. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$\overline{RM}_1^{\beta}(G) \leq n(n-1)(\Delta+1) - 2m(n-\delta) - n\Delta\delta.$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \leq t \leq n-1$.

Proof. The desired result follows from (3.10) and (3.4). □

Corollary 3.7. *Let G be a connected, non–regular graph with $n \geq 3$ vertices and m edges. Then*

$$DM_1^{\beta}(G) \leq 2m(\Delta-\delta+1) - \frac{(n\Delta-2m)^2}{\Delta ID(G)-n}.$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \leq t \leq n-1$.

Proof. The following identity is valid

$$RM_1^{\beta}(G) + DM_1^{\beta}(G) = \sum_{i=1}^{n} d_i(c_i+s_i) = 2m(\Delta-\delta+2)\,. \tag{3.11}$$

From the above identity and (3.1) we obtain the desired result. □

Corollary 3.8. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$DM_1^{\beta}(G) \leq 2m(\Delta+1) - n\Delta\delta.$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t \geq d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \leq t \leq n-1$.

Proof. The desired result follows from (3.11) and (3.4). □

Theorem 3.2. *If G is connected, non–regular graph, with $n \geq 3$ vertices and m edges, then*

$$DM_1^{\beta}(G) \geq 2m + \frac{(2m-n\delta)^2}{n-\delta ID(G)}\,. \tag{3.12}$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \leq t \leq n-1$.

Proof. For $r=2$, $p_i = \frac{s_i-1}{d_i}$, $a_i = d_i$, $i = 1,2,\ldots,n$, the inequality (2.1) becomes

$$\sum_{i=1}^{n} \frac{s_i-1}{d_i} \sum_{i=1}^{n} d_i(s_i-1) \geq \left(\sum_{i=1}^{n}(s_i-1)\right)^2\,. \tag{3.13}$$

Since

$$\begin{aligned}
\sum_{i=1}^{n} \frac{s_i-1}{d_i} &= \sum_{i=1}^{n} \frac{d_i-\delta}{d_i} = n - \delta ID(G)\,,\\
\sum_{i=1}^{n} d_i(s_i-1) &= \sum_{i=1}^{n} d_i s_i - \sum_{i=1}^{n} d_i = DM_1^{\beta}(G) - 2m\,,\\
\sum_{i=1}^{n}(s_i-1) &= \sum_{i=1}^{n}(d_i-\delta) = 2m - n\delta\,,
\end{aligned}$$

from (3.13) it follows that

$$(n-\delta ID(G)(DM_1^{\beta}(G)-2m) \geq (2m-n\delta)^2\,.$$

Since G is non–regular graph, it holds that $n - \delta ID(G) > 0$, from the above inequality we arrive at (3.12).

Equality in (3.13) holds if and only if $\Delta = d_1 = \cdots = d_t$ and $s_t = \cdots = s_n = 1$, for some t, $1 \le t \le n-1$. This implies that equality in (3.12) holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \le t \le n-1$. □

Corollary 3.9. *Let G be a connected, non–regular graph, with $n \ge 3$ vertices and m edges, then*

$$DM_1^{\alpha}(G) \ge n(\delta-1)^2 - 2m(\delta-2) + \frac{(2m-n\delta)^2}{n-\delta ID(G)} \,. \tag{3.14}$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \le t \le n-1$.

Proof. Since

$$\begin{aligned} DM_1^{\beta} - DM_1^{\alpha}(G) &= \sum_{i=1}^{n} d_i s_i - \sum_{i=1}^{n} s_i^2 = \sum_{i=1}^{n} s_i(\delta-1) = \\ &= (\delta-1)(2m - n(\delta-1))\,, \end{aligned}$$

that is

$$DM_1^{\alpha}(G) = DM_1^(\beta)(G) - (\delta-1)(2m-n(\delta-1))\,.$$

From the above identity and (3.12), the inequality (3.14) is obtained. □

Corollary 3.10. *Let G be a connected, non–regular graph, with $n \ge 3$ vertices, m edges and $\delta = 1$. Then*

$$M_1(G) \ge 2m + \frac{(2m-n)^2}{n - ID(G)} \,.$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta = 1$, for some t, $1 \le t \le n-1$.

Corollary 3.11. *Let G be a connected, non–regular graph, with $n \ge 3$ vertices and m edges. Then*

$$\overline{DM_1}^{\beta}(G) \le 2m(n-2) - n(n-1)(\delta-1) - \frac{(2m-n\delta)^2}{n-\delta ID(G)} \,.$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \le t \le n-1$.

In the following theorem we establish an upper bound for $RM_1^{\beta}(G)$ in terms of $ID(G)$ and graph parameters n, m, Δ and δ.

Theorem 3.3. *If G is connected, non–regular graph with $n \ge 3$ vertices and m edges, then*

$$RM_1^{\beta}(G) \le 2m(\Delta - \delta + 1) - \frac{(2m-n\delta)^2}{n-\delta ID(G)}. \tag{3.15}$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \le t \le n-1$.

Proof. For $r=2$, $p_i=\frac{\Delta-\delta+1-c_i}{d_i}$, $a_i=d_i$, $i=1,2,\dots,n$, the inequality (2.1) transforms into

$$\sum_{i=1}^{n}\frac{\Delta-\delta+1-c_i}{d_i}\sum_{i=1}^{n}(\Delta-\delta+1-c_i)d_i \geq \left(\sum_{i=1}^{n}(\Delta-\delta+1-c_i)\right)^2 . \tag{3.16}$$

Since

$$\sum_{i=1}^{n}\frac{\Delta-\delta+1-c_i}{d_i}=\sum_{i=1}^{n}\frac{d_i-\delta}{d_i}=n-\delta ID(G),$$
$$\sum_{i=1}^{n}(\Delta-\delta+1-c_i)d_i=2m(\Delta-\delta+1)-RM_1^{\beta}(G),$$
$$\sum_{i=1}^{n}(\Delta-\delta+1-c_i)=\sum_{i=1}^{n}(d_i-\delta)=2m-n\delta,$$

from (3.16) it follows

$$(n-\delta ID(G))(2m(\Delta-\delta+1)-RM_1^{\beta}(G))\geq(2m-n\delta)^2. \tag{3.17}$$

If G is not regular graph, then $n-\delta ID(G)>0$, therefore according to (3.17) we obtain (3.15).

If G is not regular graph, then equality in (3.16) holds if and only if $\Delta=d_1=\cdots=d_t$ and $c_{t+1}=\cdots=c_n=\Delta-\delta+1$, for some t, $1\leq t\leq n-1$. Therefore equality in (3.15) holds if and only if $\Delta=d_1=\cdots=d_t>d_{t+1}=\cdots=d_n=\delta$, for some t, $1\leq t\leq n-1$. □

Corollary 3.12. *If G is connected, not regular graph with $n\geq 3$ vertices and m edges, then*

$$RM_1^{\alpha}(G)\geq n(\Delta+1)^2-2m(2\Delta-\delta+2)+\frac{(2m-n\delta)^2}{n-\delta ID(G)}.$$

Equality holds if and only if $\Delta=d_1=\cdots=d_t>d_{t+1}=\cdots=d_n=\delta$, for some t, $1\leq t\leq n-1$.

Proof. The desired result follows from (3.7) and (3.15). □

Corollary 3.13. *If G is connected, not regular graph with $n\geq 3$ vertices and m edges, then*

$$\overline{RM}_1^{\beta}(G)\geq n(n-1)(\Delta+1)-2m(n+\Delta-\delta)+\frac{(2m-n\delta)^2}{n-\delta ID(G)}.$$

Equality holds if and only if $\Delta=d_1=\cdots=d_t>d_{t+1}=\cdots=d_n=\delta$, for some t, $1\leq t\leq n-1$.

In the next theorem we determine an upper bound on $DM_1^{\beta}(G)$ in terms of $ID(G)$ and parameters n, m and Δ.

Theorem 3.4. *Let G be a connected, non–regular graph, with $n\geq 3$ vertices and m edges, then*

$$DM_1^{\beta}(G)\leq 2m(\Delta-\delta+1)-\frac{(n\Delta-2m)^2}{\Delta ID(G)-n} . \tag{3.18}$$

Equality holds if and only if $\Delta=d_1=\cdots=d_t>d_{t+1}=\cdots=d_n=\delta$, for some t, $1\leq t\leq n-1$.

Proof. For $r=2$, $p_i=\frac{\Delta-\delta+1-s_i}{d_i}$, $a_i=d_i$, $i=1,2,\dots,n$, the inequality (2.1) becomes

$$\sum_{i=1}^{n}\frac{\Delta-\delta+1-s_i}{d_i}\sum_{i=1}^{n}(\Delta-\delta+1-s_i)d_i\geq\left(\sum_{i=1}^{n}(\Delta-\delta+1-s_i)\right)^2. \tag{3.19}$$

Since

$$\begin{aligned}\sum_{i=1}^{n}\frac{\Delta-\delta+1-s_i}{d_i} &= \sum_{i=1}^{n}\frac{\Delta-d_i}{d_i}=\Delta ID(G)-n\,,\\ \sum_{i=1}^{n}(\Delta-\delta+1-s_i)d_i &= \sum_{i=1}^{n}(\Delta-\delta+1)d_i-\sum_{i=1}^{n}d_is_i=2m(\Delta-\delta+1)-DM_1^{\beta}(G)\,,\\ \sum_{i=1}^{n}(\Delta-\delta+1-s_i) &= \sum_{i=1}^{n}(\Delta-d_i)=n\Delta-2m\,,\end{aligned}$$

from the above identities and (3.19) we obtain that

$$(\Delta ID(G)-n)(2m(\Delta-\delta+1)-DM_1^{\beta}(G))\geq(n\Delta-2m)^2\,.$$

Since G is not regular, it holds that $\Delta ID(G)-n>0$, from the above inequality we arrive at (3.18).

Equality in (3.19) holds if and only if $\Delta-\delta+1=s_1=\cdots=s_t$ and $d_{t+1}=\cdots=d_n=\delta$, for some t, $1\leq t\leq n-1$, which implies that equality in (3.18) holds if and only if $\Delta=d_1=\cdots=d_t>d_{t+1}=\cdots=d_n=\delta$, for some t, $1\leq t\leq n-1$. □

Corollary 3.14. *Let G be a connected, non–regular graph, with $n\geq 3$ vertices and m edges, then*

$$DM_1^{\beta}(G)\leq 2m(\Delta+1)-n\Delta\delta\,. \tag{3.20}$$

Equality holds if and only if $\Delta=d_1=\cdots=d_t>d_{t+1}=\cdots=d_n=\delta$, for some t, $1\leq t\leq n-1$.

Corollary 3.15. *Let G be a connected, non–regular graph, with $n\geq 3$ vertices and m edges and $\delta=1$. Then*

$$M_1(G)\leq 2m(\Delta+1)-n\Delta\,.$$

Equality holds if and only if $\Delta=d_1=\cdots=d_t>d_{t+1}=\cdots=d_n=\delta=1$, for some t, $1\leq t\leq n-1$.

Corollary 3.16. *Let G be a connected, non–regular graph, with $n\geq 3$ vertices and m edges. Then*

$$\overline{DM_1}^{\beta}(G)\geq 2m(n-\Delta+\delta-2)-n(n-1)(\delta-1)+\frac{(n\Delta-2m)^2}{\Delta ID(G)-n}\,.$$

Equality holds if and only if $\Delta=d_1=\cdots=d_t>d_{t+1}=\cdots=d_n=\delta$, for some t, $1\leq t\leq n-1$.

Corollary 3.17. *Let G be a connected, non–regular graph, with $n\geq 3$ vertices and m edges. Then*

$$\overline{DM_1}^{\beta}(G)\geq 2m(n-\Delta-2)-n((n-1)(\delta-1)-\Delta\delta)\,.$$

Equality holds if and only if $\Delta=d_1=\cdots=d_t>d_{t+1}=\cdots=d_n=\delta$, for some t, $1\leq t\leq n-1$.

Corollary 3.18. *Let G be a connected, non–regular graph, with $n \geq 3$ vertices and m edges and $\delta = 1$. Then*

$$\overline{M_1}(G) \geq 2m(n - \Delta - 2) + n\Delta\,.$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta = 1$, for some t, $1 \leq t \leq n-1$.

Corollary 3.19. *Let G be a connected, non–regular graph, with $n \geq 3$ vertices and m edge. Then*

$$DM_1^{\alpha}(G) \leq 2m(\Delta - 2\delta + 2) + n(\delta - 1)^2 - \frac{(n\Delta - 2m)^2}{\Delta ID(G) - n}\,.$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \leq t \leq n-1$.

Corollary 3.20. *Let G be a connected, non–regular graph, with $n \geq 3$ vertices and m edges and $\delta = 1$. Then*

$$M_1(G) \leq 2m\Delta - \frac{(n\Delta - 2m)^2}{\Delta ID(G) - n}\,.$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta = 1$, for some t, $1 \leq t \leq n-1$.

Corollary 3.21. *Let G be a connected, non–regular graph, with $n \geq 3$ vertices and m edges. Then*

$$DM_1^{\alpha}(G) \leq 2m(\Delta - \delta + 2) + n((\delta - 1)^2 - \Delta\delta)\,.$$

Equality holds if and only if $\Delta = d_1 = \cdots = d_t > d_{t+1} = \cdots = d_n = \delta$, for some t, $1 \leq t \leq n-1$.

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