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On Distance in Complements of Graphs

Ivan Gutman, Jia Lu, Mohamed Amine Boutiche

Abstract: Let G be a graph with n vertices and m edges. In many cases the complement of G has the following properties: it is connected, its diameter is 2, its Wiener index is equal to $\binom{n}{2}+m$, and its hyper-Wiener index is equal to $\binom{n}{2}+2m$. We characterize the graphs whose complements have the mentioned properties.

Keywords: Distance (in graph), complement (of graph), Wiener index, hyper-Wiener index

1 Introduction

In this paper we are concerned with simple graphs, that is graphs without weighted, directed, or multiple edges, and without self loops. Let G be such a graph and let $V(G)$ and $E(G)$ be, respectively, its vertex and edge sets. Let $n=|V(G)|$ and $m=|E(G)|$ be, respectively, the number of vertices and edges of G. The edge of G, connecting the vertices x and y will be denoted by xy.

The complement $\overline{G}$ of G is the graph whose vertex set is $V(G)$, and in which two vertices are adjacent if and only if they are not adjacent in G. Thus, $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$, and $|E(\overline{G})| = \binom{n}{2}-m$.

Let u and v be two vertices of the graph G. The *distance* $d_G(u,v)$ of these vertices is the length of (= number of edges in) a shortest path that connects u and v [1]. If such a path does not exist (which happens if G is not connected), then the distance between u and v is not defined.

If the graph G is connected, then the greatest distance between two of its vertices is the *diameter* of G, denoted by $diam(G)$.

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I. Gutman is with the Faculty of Science, University of Kragujevac, Serbia, and the State University of Novi Pazar, Serbia; J. Lu is with the Department of Mathematics, Central South University, Changsha 41075, P. R. China; M. A. Boutiche is with the Faculty of Mathematics, Université des Sciences et de la Technologie Houari Boumediene, El Alia, Bab Ezzouar 16111, Algeria

If the graph G is connected, then its *Wiener index* and *hyper-Wiener index* are defined as

$$W = W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) \tag{1}$$

and

$$WW = WW(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} \left[d_G(u,v) + d_G(u,v)^2 \right]. \tag{2}$$

For details on these much studied structure descriptors see the recent review [2] and the references cited therein.

In a recent paper [3], it was observed that if a graph G has the property **X** (defined below), then the Wiener index of its complement depends only on the parameters n and m, and is insensitive of any other structural details of the graphs G or $\overline{G}$. We now examine this phenomenon in more detail.

Definition 1 *We say that the graph G has property* **X** *if for every edge xy (i.e., pair of adjacent vertices x and y), there exists a vertex z, such that z is not adjacent to either x or y.*

The significance of property **X** is seen from its following (easy) consequences with regard to the complement of the graph G.

Let G be a graph with n vertices and m edges. Let $\overline{G}$ be its complement.

Proposition 1 *If the graph G has property* **X***, then $\overline{G}$ is connected.*

Proof Let x and y be two vertices of the graph G. If x and y are not adjacent in G, then they are adjacent in $\overline{G}$, and thus $d_{\overline{G}}(x,y) = 1$. If x and y are adjacent in G, then by property **X**, there is a vertex z not adjacent to either x or y. Then in $\overline{G}$, x and z are adjacent, y and z are adjacent, and x and y are not adjacent. Consequently, in $\overline{G}$, the vertices x and y are connected by a path xzy of length 2, i.e., $d_{\overline{G}}(x,y) = 2$.

Since $1 \leq d_{\overline{G}}(x,y) \leq 2$ holds for any two vertices of $\overline{G}$, this graph is connected.

□

The above proof immediately implies:

Proposition 2 *If the graph G has property* **X***, then $diam(\overline{G}) = 2$.*

Proposition 3 *If the graph G has property* **X***, then*

$$W(\overline{G}) = \binom{n}{2} + m. \tag{3}$$

Proof From the proof of Proposition 1 we known that in $\overline{G}$ there are m pairs of vertices at distance 2. The remaining $\binom{n}{2} - m$ pairs of vertices have distance 1. Therefore, by Eq. (1),

$$W(\overline{G}) = \left[\binom{n}{2} - 1\right] \times 1 + m \times 2$$

resulting in Eq. (3).

□

Proposition 4 *If the graph G has property* **X***, then*

$$WW(\overline{G}) = \binom{n}{2} + 2m\,. \tag{4}$$

Proof Repeating the arguments from the proof of Proposition 3, and bearing in mind Eq. (2), we get

$$WW(\overline{G}) = \frac{1}{2}\left(\left[\binom{n}{2} - 1\right] \times 1 + m \times 2\right) + \frac{1}{2}\left(\left[\binom{n}{2} - 1\right] \times 1 + m \times 2^2\right)$$

resulting in Eq. (4).

□

2 Graphs possessing property X

We start with the obvious:

Theorem 1 *If the graph G is disconnected, then G has property* **X**.

If G is connected, then we have:

Theorem 2 *If G is a connected graph, and* $diam(G) \geq 4$*, then G has property* **X**.

Proof If $diam(G) \geq 4$, then in $V(G)$ there must exist five vertices, say, p, q, r, s, t, such that $pq, qr, rs, st \in E(G)$, and $d_G(p,t) = 4$. Because of $d_G(p,t) = 4$, the vertex p cannot be adjacent to r, s, t, the vertex q cannot be adjacent to s, t, and the vertex r cannot be adjacent to t.

Now, for the edge pq, the vertex s satisfies the requirement of Definition 1. The same is true for the edges qr, rs, st, and the vertices t, p, and q, respectively.

Let x and y be some other pair of adjacent vertices of G. If neither x nor y is adjacent to p, then we are done. Let, therefore, x be adjacent to p. Then the vertex x cannot be adjacent to vertex t, because otherwise there would exist a path pxt of length two, which is impossible because of $d_G(p,t) = 4$. Also the vertex y cannot be adjacent to vertex t, because otherwise there would exist a path $pxyt$ of length three, which is impossible because of $d_G(p,t) = 4$. Thus, either p or t satisfy the requirement of Definition 1 with regard to the edge xy.

Therefore, all edges of G satisfy the requirement of Definition 1. Therefore, G has property **X**.

□

Theorem 2 shows that graphs possessing property **X** abound, and that most graphs encountered in any applications of graph theory possess property **X**. In additions, there exists graphs with diameter smaller than 4 that also have property **X**. Some of these are depicted in Fig. 1.

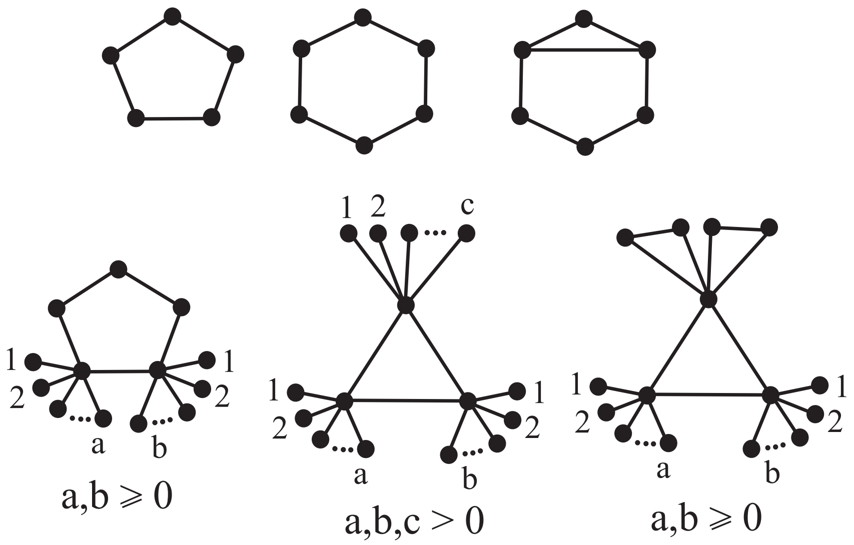

Fig. 1. Graphs with small diameter having property **X**. Except the pentagon, all other such graphs have diameter equal to 3.

3 Graphs possessing property Y

In view of Theorems 1 and 2, as well as the examples shown in Fig. 1, it appears to be more convenient to consider graphs without property **X**. These graphs must then have the following property **Y**.

Definition 2 *We say that the graph G has property* **Y** *if it has at least one edge xy (i.e., a pair of adjacent vertices x and y), such that all other vertices of G are adjacent either to x or to y or to both x and y.*

Evidently, any graph G has either property **X** or property **Y**.

According to Definition 2, the general structure of a graph $\mathscr{G}$ with property **Y** must be as follows: $\mathscr{G}$ possesses adjacent vertices x and y. Its vertex set is partitioned into three parts: $V(\mathscr{G}) = A \cup B \cup C$, where A is the set of vertices adjacent to vertex x, B is the set of vertices adjacent to vertex y, and C is the set of vertices adjacent to both x and y. Let $|A| = a$, $|B| = b$ and $|C| = c$. Without loss of generality, throughout this paper it will be assumed that $a \geq b$.

The $\mathscr{G}$-type graph with fixed values of the parameters a, b, c and with minimal number of edges has the structure shown in Fig. 2.

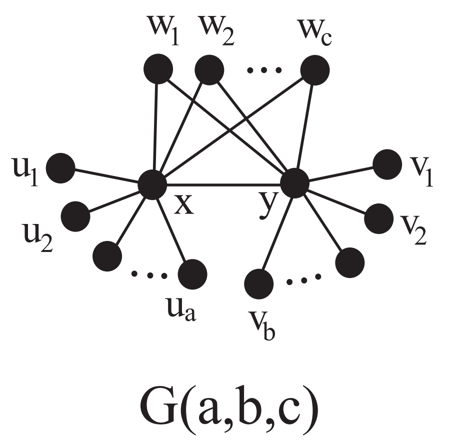

Fig. 2. The graph possessing property **Y** with minimal number of edges, for given values of the parameters $a, b, c \geq 0$.

Definition 2 immediately implies the following:

Proposition 5 *If a graph G has property* **Y***, then any graph obtained by adding one or more edges to G has property* **Y**.

The cyclomatic number of a connected graph G with n vertices and m edges is $\gamma(G) = m - n + 1$ [4]. Connected graphs with $\gamma = 1$ are the trees, whereas graphs with $\gamma = 1, 2, 3, \ldots$ are said to be unicyclic, bicyclic, tricyclic, ...

From Fig. 2 it is evident that $\gamma(G(a,b,c)) = c$. From this fact, and bearing Proposition 5 in mind, we can now characterize all graphs with property **Y** and cyclomatic number $\gamma = 0, 1, 2, \ldots$.

4 Trees with property Y

The characterization of trees (connected graphs with cyclomatic number $\gamma = 0$) with property **Y** is elementary [3]. The graph of type $G(a,b,c)$ (cf. Fig. 2) is a tree if $c = 0$. Adding more edges to $G(a,b,c)$ (cf. Proposition 5) would increase the cyclomatic number. Therefore, all trees with property **Y** are of the type $G(a,b,0)$. If $a \geq b = 0$, then this is the star. If $a \geq b > 0$ then this is the double star.

5 Unicyclic graphs with property Y

There are two classes of graphs with $\gamma = 1$, having property **Y**. The first class consists of $G(a,b,1)$ whereas the second class consists of $G(a,b,0)$ to which one new edge is added.

An edge can be added to $G(a,b,0)$ in two distinct ways. Using the notation specified in Fig. 2, the new edge may be either u_1u_2 or u_1v_1.

The structure of the three types of unicyclic graphs thus constructed is depicted in Fig. 3.

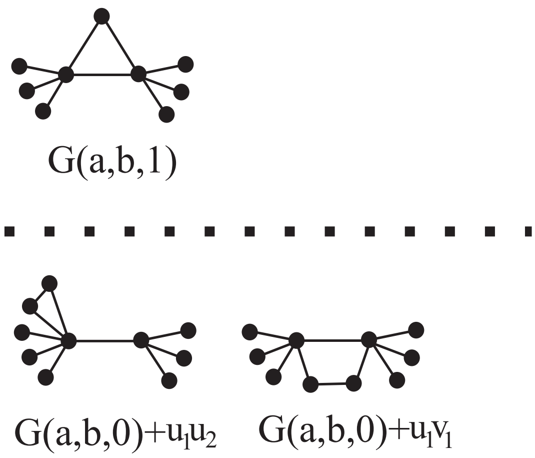

Fig. 3. All possible types of unicyclic graphs possessing property **Y**. These graphs may have any number of pendent vertices attached to the vertices *x* and *y*, including the case when there are no pendent vertices attached to *x* and/or *y*.

6 Bicyclic graphs with property Y

There are three classes of graphs with $\gamma = 2$, having property **Y**. The first class consists of $G(a,b,2)$, the second class consists of $G(a,b,1)$ to which one edge is added, the third class consists of $G(a,b,0)$ to which two edges are added.

One edge can be added to $G(a,b,1)$ in three distinct ways. Using the notation from Fig. 2, this edge may be either u_1u_2 or u_1v_1 or u_1w_1.

Two edges can be added to $G(a,b,0)$ in seven distinct way. These are $u_1u_2 + u_1u_3$, $u_1u_2 + u_3u_4$, $u_1u_2 + u_1v_1$, $u_1u_2 + u_3v_1$, $u_1u_2 + v_1v_2$, $u_1v_1 + u_2v_1$, $u_1v_1 + u_2v_2$.

The structure of the eleven (=1+3+7) types of bicyclic graphs thus constructed is depicted in Fig. 4.

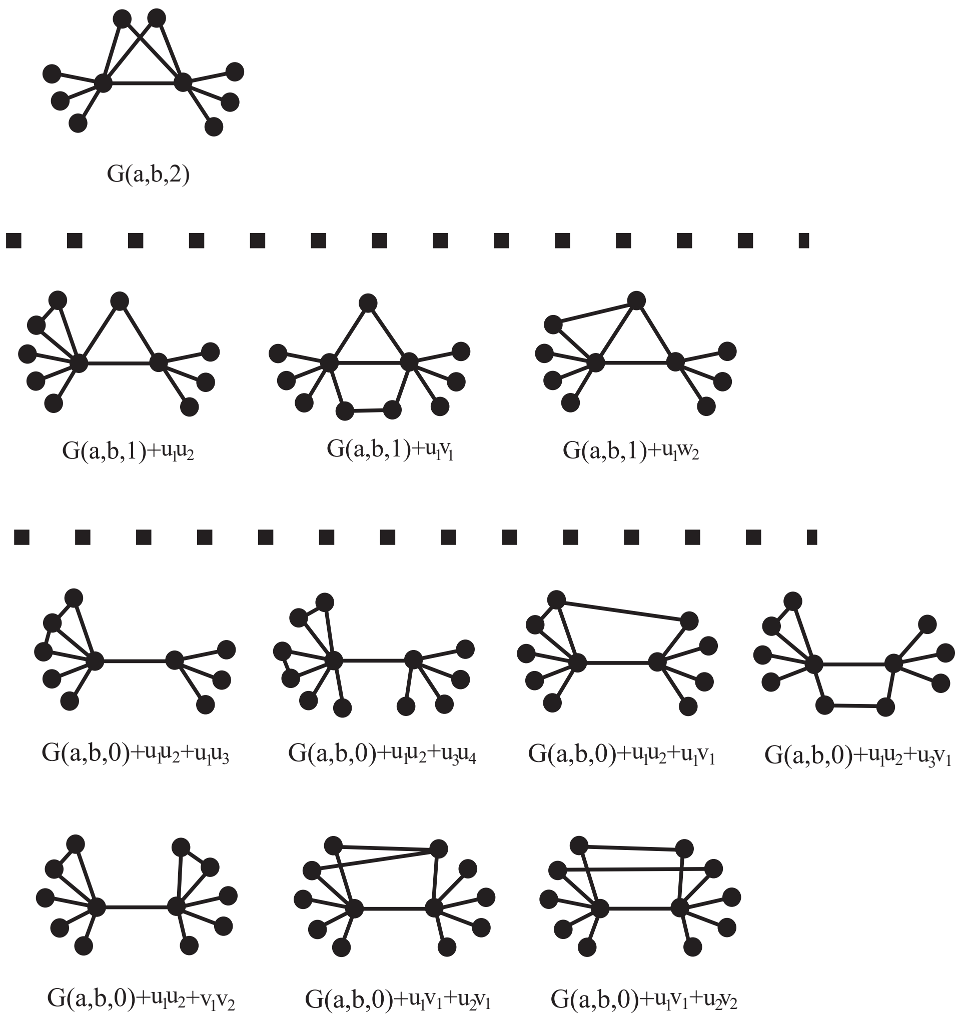

Fig. 4. All possible types of bicyclic graphs possessing property **Y**. Other details are same as in Fig. 3.

7 Concluding remarks

By means of the present analysis, we were able to characterize all graphs with cyclomatic number $\gamma = 0,1,2$, that have property $\mathbf{Y}$, i.e., those for which Eqs. (1) and (2) are violated. By this, we have extended and corrected the results of an earlier work [3].

Considerations of the same kind could be continued also for the case $\gamma = 3$ (and possibly also for $\gamma > 3$). However, the number of types of graphs that have to be distinguished becomes prohibitively large and therefore we did not pursue our study beyond $\gamma = 2$.

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